UH Stochastic analysis I, Spring 2017, Exercise-1 (27.1 and 1.2 2017)

In all exercises $(B_t : t \geq 0)$ is a Brownian motion on the probability $(\Omega, \mathcal{F}_t, P)$, in the filtration $\mathbb{F} = (\mathcal{F}_t : t \geq 0)$, which means that B is $\mathbb{F}$-adapted and for $0 \leq s \leq t$, $(B_t - B_s) \perp\!\!\!\perp \mathcal{F}_s$ w.r.t. P.

1. (Computer exercise)

   (a) Use the `octave` function `levybm.m` downloadable from the course web-page to plot some paths of the Brownian motion sampled by using Paul Levy construction.

   (b) write an `octave` program plotting a path of the two dimensional Brownian motion $\mathbf{B}_t = (B_t^{(1)}, B_t^{(2)})$ with values in $\mathbb{R}^2$ on the time interval $[0, 1]$. Here $B_t^{(1)}$ and $B_t^{(2)}$ are two independent Brownian motions.

   If you need to install the `octave` scientific programming language, you can download it from `https://www.gnu.org/software/octave/`

2. Consider a standard Gaussian random variable $G(\omega)$ with probability density

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\bigl(-x^2/2\bigr)$$

   (a) Check that

$$\frac{d}{dx}\phi(x) = -x\phi(x)$$

   (b) Compute

$$E_P(\exp(\lambda G^2)\bigr), \quad \lambda \in \mathbb{R}$$

   (c) Use Fubini to prove following *Gaussian integration by parts formula*: if

$$f(x) = f(0) + \int_0^x f'(t)dt$$

   with $E_P(|f'(G)|) < \infty$, then

$$E_P\bigl(f'(G)\bigr) = E_P\bigl(f(G)G\bigr)$$

(d) Use the Gaussian integration by parts formula to compute by induction the Gaussian moments

$$E_P(G^{2n}) = \frac{(2n)!}{n!2^n}$$

3. (a) Show that

$$(t,x) \mapsto p(t;x,y) := \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}$$

solves the partial differential equation

$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2 p}{\partial x^2}\,.$$

(b) Show that

$$(t,x) \mapsto p^{(\mu)}(t;x,y) := \frac{1}{\sqrt{2\pi t}} e^{-(x-y-\mu t)^2/2t}$$

solves

$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2 p}{\partial x^2} + \mu\frac{\partial p}{\partial x}.$$

For which equation the function

$$(t,y) \mapsto p^{(\mu)}(t;x,y)$$

is a solution?

4. Let B_t be a Brownian motion.

(a) Let $c > 0$. Show that $B_t^{(c)} = (B_{t+c} - B_c)$, $t \geq 0$ is a Brownian motion in its own filtration.

(b) Show that for $c > 0$, $\widehat{B}_t := c^{-1/2} B_{ct}$, $t \geq 0$ is a Brownian motion in its own filtration. We say that B is self-similar of index $\alpha = 1/2$. Does this transformation preserve the law of the Poisson process ?

(c) Show that $\check{B}_t := t B_{1/t}$ $t > 0$ is a Brownian motion (in its own filtration), Does this transformation preserve the law of the Poisson process ?

(d) Show that $\lim_{t\downarrow 0} \check{B}_t = 0$ P-almost surely.

Hint: use Chebychev inequality together with the Borel Cantelli lemma.

5. Let $(\dot\eta_n(t) : n \in \mathbb{N})$ an orthonormal system of functions in $L^2([0,1], dt)$, with

$$\int_0^1 \dot\eta_n(s)\dot\eta_m(s)ds = \delta_{n,m}$$

for example the Haar system we have used in Paul Lévy's construction of Brownian motion, and let $(G_n(\omega) : n \in \mathbb{N})$ a sequence of i.i.d. Gaussian random variables with $E(G) = 0$ and $E(G^2) = 1$. show that the random functions

$$X_t^{(n)}(\omega) = \sum_{k=1}^n \dot\eta_k(s) G_k(\omega)$$

do not converge in $L^2(\Omega \times [0,1], dP \otimes dt)$. Hint: compute the squared norm

$$E\biggl(\int_0^t \biggl\{\sum_{k=1}^n G_d(\omega)\dot\eta_k(s)\biggr\}^2 ds\biggr)$$

The next 3 exercises are meant to refresh your background in discrete time martingale theory.

6. In discrete time, let M_t be a martingale with respect to P and the filtration $(\mathcal{F}_t : t \in \mathbb{N})$, such that $M_0 = 0$ and $E_P(M_t^2) < \infty\ \forall t$. Show that

$$\text{Variance}(M_t) = \sum_{s=1}^t E_P\bigl(\{M_s - M_{s-1}\}^2\bigr)$$

Check that the same result holds in continuous time.

7. Consider a symmetric random walk on $\mathbb{Z}$ in discrete time,

$$M_n = X_1 + \cdots + X_n$$

where $(X_k : k \in \mathbb{N})$ are independent and identically distributed Bernoulli random variables with $P(X_n = 1) = P(X_n = -1) = 1/2$.

We use the filtration generated by the random walk, $\mathbb{F} = \bigl(\mathcal{F}_n^X\bigr)$, with $\mathcal{F}_n^X = \sigma(X_k : 0 \le k \le n)$.

(a) Consider the stopping time $\sigma(\omega) = \min(\tau_a, \tau_b)$ where $a < 0 < b \in \mathbb{N}$, and the stopped martingales $(M_{t\wedge\sigma})_{t\in\mathbb{N}}$ and $(M^2_{t\wedge\sigma} - t \wedge \sigma)_{t\in\mathbb{N}}$. Show that Doob's martingale convergence theorem applies and

$$\lim_{t\to\infty} M_{t\wedge\sigma}(\omega) = M_\sigma(\omega)$$

exists P-almost surely.

(b) Consider now $(M^2_{t\wedge\sigma} - t\wedge\sigma)$. Use the martingale property together with the reverse Fatou lemma to show that $E(\sigma) < \infty$ which implies $P(\sigma < \infty) = 1$.

(c) For $a < 0 < b \in \mathbb{N}$, compute $P(\tau_a < \tau_b)$.
Hint: a martingale has constant expectation $E_P(M_t) = E_P(M_0)$. This holds also for the stopped martingale $M^\tau_t = M_{t\wedge\tau}$.

8. Suppose we have an urn which contains at time $t = 0$ two balls, one black and one white. At each time $t \in N$ we draw uniformly at random from the urn one ball, and we put it back together with a new ball of the same colour.

We introduce the random variables
$X_t(\omega) = \mathbf{1}\{$ the ball drawn at time t is black $\}$
and denote $S_t = (1 + X_1 + \cdots + X_t)$,
$M_t = S_t/(t + 2)$, the proportion of black balls in the urn.
We use the filtration $\{\mathcal{F}_n\}$ with $\mathcal{F}_n = \sigma\{X_s : s \in \mathbb{N}, s \le t\}$.

i) Compute the Doob decomposition of (S_t), $S_t = S_0 + N_t + A_t$, where (N_t) is a martingale and (A_t) is predictable.

ii) Show that (M_t) is a martingale and find the representation of (M_t) as a martingale transform $M_t = (C \cdot N)_t$, where (N_t) is the martingale part of (S_t) and (C_t) is predictable.

iv) Note that the martingale $(M_t)_{t\ge0}$ is uniformly integrable (Why ?). Show that P a.s. and in L^1 exists $M_\infty = \lim_{t\to\infty} M_t$. Compute $E(M_\infty)$.

v) Show that $P(0 < M_\infty < 1) > 0$.

Since $M_\infty(\omega) \in [0, 1]$, it is enough to show that $0 < E(M_\infty^2) < E(M_\infty)$ with strict inequalities.

Hint: compute the Doob decomposition of the submartingale (M_t^2), and than take expectations before going to the limit to find the value of $E(M_\infty^2)$.