Department of Mathematics and Statistics
Real Analysis I
Exercise 7
3-4.5.2017

1. Let $E \subset \mathbb{R}$ be Lebesgue-measurable and $m(E) > 0$. Show that
$$E - E = \{a - b : a, b \in E\}$$
contains an interval.

2. (a) Let $G \subset \mathbb{R}^n$ be an open set. Show that every point in G is a density point of G.
   (b) Demonstrate with the help of an example that a point in the complement of an open set G can be a density point of G.
   (c) Construct a subset A of $\mathbb{R}$, such that A is not open, even though every point $x \in A$ is a density point of A.

3. Let $\mathbb{Q} = \{q_k \colon k \in \mathbb{N}\}$ be the set of all rational numbers. Define
$$f(x) = \sum_{\substack{k \in \mathbb{N} \\ q_k \leq x}} 2^{-k}, \quad x \in \mathbb{R}.$$
Prove:
   (a) $f \colon \mathbb{R} \to (0, 1)$ is strictly increasing.
   (b) f is continuous at $x \iff x \in \mathbb{R} \setminus \mathbb{Q}$.

4. (a) Exhibit a (simple) example of a function $f \colon \mathbb{R} \to \mathbb{R}$, which has bounded variation, but is not continuous.
   (b) Let $f \colon [0, 1] \to \mathbb{R}$,
$$f(x) = \begin{cases} x \sin \frac{1}{x}, & \text{if } 0 < x \leq 1, \\ 0, & \text{if } x = 0. \end{cases}$$
   Show that f is continuous, but that f does not have bounded variation.

5. The function $f \colon [a, b] \to \mathbb{R}$ is a Lipschitz-function, if there is a constant $L < \infty$, so that $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in [a, b]$. Show:
   (a) every continuously differentiable function $f \colon [a, b] \to \mathbb{R}$ is Lipschitz,
   (b) every Lipschitz-function has bounded variation,
   (c) every Lipschitz-function is differentiable a.e.

6. Let $f \colon [a, b] \to \mathbb{R}$ be of bounded variation. Denote $V(x) = V_f(a, x)$. Prove that f is continuous $\iff$ V is continuous.