Department of Mathematics and Statistics
Real Analysis I
Exercise 6
26-27.4.2017

1. Let (X, Γ, μ) be a measure space, $f \in L^p(X)$ and $t > 0$. Show that *Chebyshev's inequality* holds:
$$\mu\big(\{x \in X \colon |f(x)| > t\}\big) \leq \left(\frac{\|f\|_p}{t}\right)^p .$$

2. We say that a measurable function $f\colon \mathbb{R}^n \to \dot{\mathbb{R}}$ belongs to the "weak L^1-space" weak-$L^1(\mathbb{R}^n)$ if there is a constant $c = c_f < \infty$, so that
$$m\big(\{x \in \mathbb{R}^n \colon |f(x)| > t\}\big) \leq \frac{c}{t} \quad \forall t > 0.$$
   (a) Verify that $L^1(\mathbb{R}^n) \subset$ weak-$L^1(\mathbb{R}^n)$.
   (b) Show with the help of an example that weak-$L^1(\mathbb{R}) \not\subset L^1(\mathbb{R})$.

3. Let $A,\ B \subset \mathbb{R}^n$ be measurable sets, so that
$$m\big(A \cap B^n(x, 1/i)\big) \leq m\big(B \cap B^n(x, 1/i)\big)$$
for all $x \in A$ and $i \in \mathbb{N}$, where $B^n(x, 1/i) = \{y \in \mathbb{R}^n : |y - x| < 1/i\}$. Show that $m(A) \leq m(B)$.

4. Let $f \in L^1(\mathbb{R}^n)$. Show that $Mf(x) < +\infty$ for a.e. $x \in \mathbb{R}^n$.

5. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ be a locally integrable function, $t > 0$, $A_t = \{x \in \mathbb{R}^n \colon |f(x)| \leq t/2\}$, $g = f\chi_{A_t}$ and $h = f - g$. Show that
$$\{x \in \mathbb{R}^n \colon Mf(x) > t\} \subset \{x \in \mathbb{R}^n \colon Mh(x) > t/2\}.$$

6. Let $f\colon \mathbb{R}^n \to \dot{\mathbb{R}}$ be measurable. Show that for any $t > 0$ one has:
$$m\big(\{x \in \mathbb{R}^n \colon Mf(x) > t\} \leq \frac{2 \cdot 5^n}{t} \int_{\{x\colon |f(x)| > t/2\}} |f(y)| dy.$$