Department of Mathematics and Statistics
Real Analysis I
Exercise 5
12.4.2017, 20.4.2017

1. Prove that one can not, in general, choose the set F in Egorov's theorem so that $\mu(X \setminus F) = 0$.

2. Give an example of functions $f \in L^1(\mathbb{R})$ and $g \in L^1(\mathbb{R})$ such that the function
$$y \mapsto f(x-y)g(y)$$
is not integrable for all $x \in \mathbb{R}$.

3. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be continuous and $g_\varepsilon = m\big(B(0,\varepsilon)\big)^{-1}\chi_{B(0,\varepsilon)}$. Prove that $g_\varepsilon * f(x) \to f(x)$ as $\varepsilon \to 0+$.

4. Suppose that $f \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, and $g \in L^1(\mathbb{R}^n)$. Prove that $f * g(x)$ exists for a.e. $x \in \mathbb{R}^n$, $f * g \in L^p(\mathbb{R}^n)$, and
$$\|f * g\|_p \le \|f\|_p \|g\|_1.$$

5. Let $g \ge 0$ be a measurable function in $\mathbb{R}^n$. Suppose that there exists a constant C such that $\|f * g\|_p \le C\|f\|_p$ for every non-negative $f \in L^p(\mathbb{R}^n)$. Prove that $C \ge \|g\|_1$.

6. Prove the following covering theorem: Let $\{B(x_i, r_i)\}, i \in I$, be a finite collection of balls in a metric space (X, d). Then there exists $J \subset I$ such that the balls $B(x_i, r_i), i \in J$, are mutually disjoint and
$$\bigcup_{i\in I} B(x_i, r_i) \subset \bigcup_{i \in J} B(x_i, 3r_i).$$