Department of Mathematics and Statistics
Real analys I
Exercise 4
5-6.4.2017

1. Let (X, Γ, μ) be a complete measure space, $1 \leq p < \infty$, and $f_i \in L^p(X)$, $i \in \mathbb{N}$. Assume that f is a measurable function so that $\|f_i - f\|_p \to 0$. Show that there is a subsequence (f_{i_k}) of the sequence (f_i) so that $f_{i_k} \to f$ almost everywhere.

2. Suppose that $\mu(X) < \infty$. Prove that
$$\|f\|_\infty = \lim_{p\to\infty} \|f\|_p.$$
for all $f \in L^\infty(\mu)$. Is the assumption $\mu(X) < \infty$ necessary (and why)?

3. Let $C(I)$, $I = [0, 1]$, be the vector space of all continuous functions $f : I \to \mathbb{R}$.
   (a) Prove that $C(I) \subset L^p(I, m_1)$ for all $1 \leq p \leq \infty$.
   (b) Prove that $C(I)$ is complete with respect to the norm $\|\cdot\|_\infty$.
   (c) Prove by a counterexample that $C(I)$ is not complete with respect to the norm $\|\cdot\|_p$ for $1 \leq p < \infty$.

4. Let (X, Γ, μ) be a complete measure space and $1 \leq p, q < \infty$.
   (a) Suppose that $f_i \in L^p$, $i \in \mathbb{N}$, $\|f_i - f\|_p \to 0$ and $f_i \to g$ a.e. Show that $f = g$ a.e.
   (b) Suppose that $f_i \in L^p \cap L^q$, $i \in \mathbb{N}$, $\|f_i - f\|_p \to 0$ and $\|f_i - g\|_q \to 0$. Show that $f = g$ a.e.

5. Let $1 \leq p < \infty$ and $f, f_k \in L^p(X)$, $k \in \mathbb{N}$. Suppose that
$$f_k \to f \text{ a.e. and } \|f_k\|_p \to \|f\|_p.$$
Prove that $\|f_k - f\|_p \to 0$, in other words, $f_k \to f$ in $L^p(X)$.

6. Let $1 < p < \infty$ and $q = \frac{p}{p-1}$. We say that a sequence (u_j), $u_j \in L^p$, *converges weakly in* L^p to a function $u \in L^p$ if
$$\int_X u_j g \, d\mu \to \int_X ug \, d\mu \text{ as } j \to \infty$$
for all $g \in L^q$.
   (a) Prove that $f_j \to f$ weakly in L^p if $f_j \to f$ in L^p.
   (b) Let $X = [0, 1]$ and $u_j = j^{1/p}\chi_{[0,1/j]}$ for $j = 1, 2, \ldots$. Prove that (u_j) converges to 0 weakly in L^p but not in L^p.