Department of Mathematics and Statistics
Real analys I
Exercise 3
29-30.3.2017

1. Let (X, Γ, μ) be a complete measure space and let $f \in L^\infty(X)$. Show that
$$\|f\|_\infty = \inf\big\{\sup\{|f(x)| \colon x \in X \setminus N\} \colon N \in \Gamma, \mu(N) = 0\big\}.$$

2. Let $\mu(X) < \infty$ and $1 \leq q < p < \infty$.
   (a) Show *without using* Hölder's inequality that $L^p(\mu) \subset L^q(\mu)$.
   (b) Show (by using Hölder's inequality) that
   $$\|f\|_q \leq \|f\|_p \,(\mu(X))^{(p-q)/pq}\,,$$
   for $f \in L^p(\mu)$.

3. Show that $L^p(\mu) \cap L^q(\mu) \subset L^r(\mu)$ if $1 \leq p \leq r \leq q \leq \infty$.

4. Let X be an arbitrary set, $\Gamma = \mathcal{P}(X)$, and μ the counting measure on X, that is, $\mu(A) = \#A$ for $A \subset X$. [In this case we denote $\ell^p(X) = L^p(X, \mu)$.] Let $1 \leq q \leq p \leq \infty$. Show that $\ell^q(X) \subset \ell^p(X)$ and
$$\|f\|_\infty \leq \|f\|_p \leq \|f\|_q \leq \|f\|_1.$$
Compare this fact with Theorem 1.33.

5. Suppose that $f_j \to f$ in $L^p(\mathbb{R}^n)$ and $g_j \to g$ in $L^q(\mathbb{R}^n)$, where $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Show that $f_j g_j \to fg$ in $L^1(\mathbb{R}^n)$.

6. Let (X, Γ, μ) be a measure space, $1 \leq p \leq q < \infty$, and $A \in \Gamma$ a measurable set so that $0 < \mu(A) < \infty$. Show that
$$\left(\frac{1}{\mu(A)} \int_A |f|^p \, d\mu\right)^{1/p} \leq \left(\frac{1}{\mu(A)} \int_A |f|^q \, d\mu\right)^{1/q}$$
for all $f \in L^q(A)$.