Department of Mathematics and Statistics
Real analys I
Exercise 2
22-23.3.2017

1. Let (X, Γ, μ) be a neasure space. Suppose that $(X, \mathcal{F}, \nu)$ is a complete measure space (i.e. μ is a complete measure) such that $\Gamma \subset \mathcal{F}$ and $\nu|\Gamma = \mu$. Prove that $\bar{\Gamma} \subset \mathcal{F}$ and $\nu|\bar{\Gamma} = \bar{\mu}$.

2. Let $\mu = m_n|\operatorname{Bor}\mathbb{R}^n$. Construct an example of a sequence of Borel functions $f_j\colon \mathbb{R}^n \to \mathbb{R},\ j \in \mathbb{N}$, and a function $f\colon \mathbb{R}^n \to \mathbb{R}$ such that $f_j(x) \to f(x)$ μ-a.e. x as $j \to \infty$ and that f is not a Borel function.

3. Find a sequence $p = (p_1, p_2, \ldots)$ so that the corresponding Cantor set $C(p)$ has measure (1-dimensional Lebesgue measure) $\frac{1}{4}$.

4. Let (f_j) be a sequence of nonnegative Lebesgue-measurable functions $f_j\colon \mathbb{R} \to \mathbb{R}$ such that
$$\int_{\mathbb{R}} f_j\, dm \to 0.$$
Is it true that necessarily $f_j(x) \to 0$ for a.e. $x \in \mathbb{R}$? [Why??]

5. Let $f\colon B^n(0,1) \to \dot{\mathbb{R}},\ f(x) = \log|x|$. For which values of $p \in [1, \infty)$ does it hold that
$$\int_{B^n(0,1)} |f|^p dm < \infty?$$

6. Construct examples of integrable functions (over $\mathbb{R}$) $f_i, f\colon \mathbb{R} \to \mathbb{R},\ i \in \mathbb{N}$, so that $f_i \to f$ uniformly, but
$$\int_{\mathbb{R}} |f_i - f| dm \not\to 0.$$