Department of Mathematics and Statistics
Real Analysis I
Exercise 1
15-16.3.2017

You may gain extra credit points by solving these exercises. More precisely, these are counted to the numerator only when calculating the percentage for extra credit points.

1. Let (X, Γ, μ) be a measure space and $E \in \Gamma$. Prove that
   (a) $\Gamma_E = \{A \in \Gamma \colon A \subset E\}$ is a σ-algebra in E and
   (b) $\mu | \Gamma_E$ is a measure in E.

2. Let X be a nonempty set. Define $\mu^* \colon \mathcal{P}(X) \to \{0, 1\}$ by setting $\mu^*(\emptyset) = 0$ and $\mu^*(A) = 1$ if $A \neq \emptyset$. Prove that μ^* is an outer measure. What are the μ^*-measurable sets?

3. Let $I = [0,1] \times [0,1]$ and $f(x,y) = (x-y)/(x+y)^3$ when $(x,y) \in I \setminus \{(0,0)\}$ and $f(0,0) = 0$. Compute the integrals
$$\int_0^1 \left(\int_0^1 f(x,y)\, dy \right) dx \quad \text{ja} \quad \int_0^1 \left(\int_0^1 f(x,y)\, dx \right) dy.$$
   Is f integrable in I?

4. Let $f \colon (0,1) \times (1, \infty) \to \mathbb{R}$,
$$f(x,y) = e^{-xy} - 2e^{-2xy}.$$
   Prove that
$$\int_0^1 \left(\int_1^\infty f(x,y)\, dy \right) dx \neq \int_1^\infty \left(\int_0^1 f(x,y)\, dx \right) dy$$
   and comment on Fubini's theorem.

5. For each $A \subset \mathbb{R}^n$ define
$$\tilde{A} = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \colon x - y \in A\}.$$
   Prove that $m_{2n}(\tilde{A}) = 0$ if $m_n(A) = 0$.

6. Prove that $B \subset \mathbb{R}$ is a Borel set if and only if it is a Borel set also as a subset of the plane. More precisely,
$$\operatorname{Bor} \mathbb{R} = \{B \subset \mathbb{R} \colon \{(x,0) \colon x \in B\} \in \operatorname{Bor} \mathbb{R}^2\}.$$
   Is there a similar relation between σ-algebras $\operatorname{Leb} \mathbb{R}$ and $\operatorname{Leb} \mathbb{R}^2$?