PDEs II
Demo 2

1. Let α be a constant such that $0 < \alpha < 1$. Define function $u : B(0,1) = \{y \in \mathbb{R}^2 : |y| < 1\} \to \mathbb{R}$ as

$$u(x) = |x|^{\alpha-1}x_1, \quad \text{for} \quad x = (x_1, x_2) \in B_1.$$

For which $1 \leq q \leq \infty$ does u belong to $W^{1,q}(B(0,1))$?

2. Prove that if $u \in W^{1,p}((0,1))$ for some $1 \leq p < \infty$, then

$$|u(x) - u(y)| \leq |x-y|^{1-\frac{1}{p}} \left(\int_0^1 |u'|^p \, dt \right)^{\frac{1}{p}}$$

for all $x, y \in (0,1)$.

3. Denote by U the open square $\{x \in \mathbb{R}^2 : |x_1| < 1, |x_2| < 1\}$. Define

$$u(x) = \begin{cases} 1 - x_1 & x_1 > 0, |x_2| < x_1; \\ 1 + x_1 & x_1 < 0, |x_2| < -x_1; \\ 1 - x_2 & x_2 > 0, |x_1| < x_2; \\ 1 - x_2 & x_2 < 0, |x_1| < -x_2. \end{cases}$$

For which $1 \leq q \leq \infty$ does u belong to $W^{1,q}(U)$?

4. Integrate by parts to prove

$$\int_\Omega |Du|^2 \, dx \leq \left(\int_\Omega u^2 \, dx \right)^{\frac{1}{2}} \left(\int_\Omega |D^2 u|^2 \, dx \right)^{\frac{1}{2}}$$

for all $u \in C_0^\infty(\Omega)$.

5. Integrate by parts to prove

$$\int_\Omega |Du|^p \, dx \leq C \left(\int_\Omega |u|^p \, dx \right)^{\frac{1}{2}} \left(\int_\Omega |D^2 u|^p \, dx \right)^{\frac{1}{2}}$$

for $2 \leq p < \infty$ and all $u \in C_0^\infty(\Omega)$, where $C = C(n,p) > 0$.

6. Verify that

$$u(x) = \log\log\left(1 + \frac{1}{|x|}\right)$$

belongs to $W^{1,n}(B(0,1)), B(0,1) \subset \mathbb{R}^n$.