PDE II
Demo 1

1. Let $1 < p < \infty$ and $p' = p/(p-1)$. Show that for all $a, b \geq 0$ and $\varepsilon > 0$,

$$\text{i)} \quad ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'};$$
$$\text{ii)} \quad ab \leq \varepsilon a^p + C(\varepsilon) b^{p'},$$

where $C(\varepsilon) = (\varepsilon p)^{-\frac{1}{p-1}}/p'$.

2. Let $1 < p < \infty$ and $p' = p/(p-1)$. Let $\Omega \subset \mathbb{R}^n$ be an open set. Suppose that $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$. Show that

$$\int_\Omega |uv|\, dx \leq \left(\int_\Omega |u|^p\, dx \right)^{\frac{1}{p}} \left(\int_\Omega |v|^{p'}\, dx \right)^{\frac{1}{p'}}.$$

3. Let $A \in \mathbb{S}^{n \times n}$ be a symmetric matrix. Suppose that $A \geq 0$, that is, all of the eigenvalues of A are nonnegative. Show that for all $\xi, \eta \in \mathbb{R}^n$,

$$(A\xi, \eta) \leq (A\xi, \xi)^{\frac{1}{2}} (A\eta, \eta)^{\frac{1}{2}}.$$

4. Let $\Omega \subset \mathbb{R}^n$ be an open set and $u \in L^2(\Omega)$. Suppose that

$$\int_\Omega u\varphi\, dx = 0$$

for all $\varphi \in C_0^\infty(\Omega)$. Show that $u(x) = 0$ for almost every $x \in \Omega$ (with respect to the Lebesgue measure).

5. Let α be a constant such that $0 < \alpha < 1$. Define the function $u : B(0,1) = \{y \in \mathbb{R}^2 : |y| < 1\} \to \mathbb{R}$ as

$$u(x) = |x|^{\alpha - 1} x_1, \quad \text{for} \quad x = (x_1, x_2) \in B_1.$$

Define

$$A(x) = \begin{pmatrix} \frac{x_1^2 + \alpha^2 x_2^2}{|x|^2} & (1-\alpha^2)\frac{x_1 x_2}{|x|^2} \\ (1-\alpha^2)\frac{x_1 x_2}{|x|^2} & \frac{\alpha^2 x_1^2 + x_2^2}{|x|^2} \end{pmatrix}.$$

Show that

i) $\alpha^2 |\xi|^2 \leq (A(x)\xi, \xi) \leq |\xi|^2, \quad \forall x \in B_1, \xi \in \mathbb{R}^2$.

ii) u is a solution of

$$\operatorname{div}(A(x)\nabla u) = 0 \quad \text{in } B(0,1) \setminus \{0\}.$$