Model theory
Exercise 9

1. Choose g so that for all $i<\alpha$,

~~[illegible]~~ $\mathfrak{D} \models g_i(g(a_i), g(a_{j_0}), .., g(a_{j_m}))$

where $g_i(v_1, a_{j_0}, .., a_{j_m})$ isolates

$t(a_i / A \cup \{a_j \mid j<i\})$.

2. Let $\mathfrak{A} = \mathbb{Q}$

3. It is enough to find $\mathfrak{A}_0, \mathfrak{A}_1 \models T_{rgr}$ s.t.

(i) ~~[illegible]~~ $(X, \emptyset) \subseteq \mathfrak{A}_0, \mathfrak{A}_1$

(ii) There is no $X \subseteq \mathfrak{A} \preceq \mathfrak{A}_0$ and elem. embedding $g : \mathfrak{A} \to \mathfrak{A}_1$ s.t. $g \restriction X = id$.

To guarantee (ii), it is enough to construct $\mathfrak{A}_0$ s.t. for all $a \in \mathfrak{A}_0 \setminus X$, $(a,x) \in E$ for only finitely many $x \in X$ and for all $a \in \mathfrak{A}_1 \setminus X$, $(a,x) \notin E$ for only finitely many $x \in X$.

We construct $\mathfrak{A}_0$, $\mathfrak{A}_1$ can be constructed similarly.

$\mathfrak{A}_0 = \bigcup_{i<\omega} A_i$ where

$A_0 = X$ and A_{i+1} is got as follows:

Let Y_j, $j<\omega$, list all finite subsets of A_i

and let a_j, $j<\omega$ be new elements.

then $dom(A_{i+1}) = dom(A_i) \cup \{a_j \mid j < \omega\}$
and E is defined so that
$A_i \subseteq A_{i+1}$ (as structures) and
for all $x \in A_i$, $(a_j, x) \in E$ if
$x \in Y_j$ and for all $j, k < \omega$
$(a_j, a_k) \notin E$.

4. Show first that it is enough to find $\mathfrak{A} \preceq \mathfrak{L}$ and $b_i \in \mathfrak{L}$, $i \in J$ s.t. $(b_i)_{i \in J}$ is indiscernible over A and for all $i_0 < \ldots < i_n \in J$ and $a \in A^m$, $\mathfrak{L} \models \varphi(b_{i_0}, \ldots, b_{i_n}, a)$ iff there are $j_0 < \ldots < j_n \in I$ s.t. $\mathfrak{A} \models \varphi(a_{j_0}, \ldots, a_{j_n}, a)$.
Then use compactness.

5. If $a_i \in acl(A \cup \{a_j \mid j < i\})$ then a_i is a root of a polynomial P over the ring generated by $A \cup \{a_j \mid j < i\}$.
~~Sxxx~~ But then every a_k, $k > i$, is a root of this polynomial P. Since P has only finitely many roots, we have a contradiction.

For the other direction, apply Fact 6.2 (iv) inductively (and 6.2 (ii)).