Model theory
Exercise 8

1. Let $\mathfrak{A} = \prod_{i<\omega} \mathfrak{A}_i / U$, $A = \{ f_i/U \mid i<\omega \} \subseteq \mathfrak{A}$,

$p = \{ \varphi_i(v_1, f_0/U, \dots, f_i/U) \mid i<\omega \} \in S_1(A)$

We need to find $f \in \prod_{i<\omega} \mathfrak{A}_i$ s.t.

f/U realizes p.

Choose f s.t. for all $n<\omega$,

$f(n)$ realizes the type $\{ \varphi_i(v_1, f_0/U, \dots, f_i/U) \mid i<m_n \}$

where m_n is the largest $m \leq n$ s.t.

$\mathfrak{A}_n \models \exists v_1 \bigwedge_{i<m} \varphi_i(v_1, f_0(n), \dots, f_i(n))$.

2. Let $p' = p$ be as in remark, $L = \{+, \times, 0, 1, -\} \cup \{ c_i \mid i<\omega_1 \}$

and $T = T_{f_0} \cup \{ \neg c_i = c_j \mid i<j<\omega_1 \}$.

Then no model of T omit p'.

To see that T locally omits p',

assume that this is not the case.

It follows that there are ~~finite~~ countable $T' \subseteq T$ and

$\varphi(v_1)$ s.t. $T' \vdash \forall v_1 (\varphi \to \psi)$ and $T \nvdash \forall v_1 \neg\varphi(v_1)$

all $\psi \in p'$. Let $\mathfrak{A}$ be the field

of complex numbers with interpretations

for c_i, $i<\omega_1$, s.t. $\mathfrak{A} \models T$ and

~~[illegible]~~

$\mathfrak{A} \models \exists v_1 \varphi$. Let $\mathfrak{D} = acl(\emptyset) \preceq \mathfrak{A} \restriction \{+, \times, 0, 1, -\}$

Then we can interpret the constants that appear in T' or φ in such a way in $\mathfrak{D}$ that $\mathfrak{D} \models T'$ and $\mathfrak{D} \models \exists v_1 \varphi$.

But then $\mathfrak{D}$ realizes p' $\lightning$.

3. Notice that if φ is not implied by any complete fml,

(i) There are φ_0, φ_1 s.t. $\varphi_0 \wedge \varphi_1$ is contradictory and $T \vdash \forall v_1 .. \forall v_n (\varphi_i \rightarrow \varphi)$

(ii) If $T \vdash \forall v_1 .. \forall v_n (\psi \rightarrow \varphi)$ then ψ is not implied by any complete fml.

~~Thus for all [illegible]~~

~~we can construct~~ Thus for all $\eta : n \rightarrow 2$, $n < \omega$, one can construct φ_η s.t. for all η

(a) $\varphi_{\eta \frown (0)} \wedge \varphi_{\eta \frown (1)}$ is contradictory

(b) ~~[illegible]~~ $T \vdash \forall v_1 .. \forall v_n (\varphi_{\eta \frown (i)} \rightarrow \varphi_\eta)$

(c) $\varphi_\emptyset = \varphi$.

Then for all $\eta : \omega \rightarrow 2$ $p_\eta = \{\varphi_{\eta \restriction n} \mid n < \omega\}$ are consistent pairwise contradictory types.

4. From the proof of Exercise 7.2 it follows that $|S_n(\emptyset)|$ is countable for all $n \in \omega \setminus \{0\}$.

5. Let $T = Th(\mathfrak{A})$ where $\mathfrak{A} = (2^\omega, P_\eta^{\mathfrak{A}})_{\eta \in 2^{<\omega}}$, where $\xi \in P_\eta^{\mathfrak{A}}$ if $\eta \subseteq \xi$.

6. Let $\mathfrak{A}$ be the field of complex numbers. Then $t(n/\emptyset)$, $n \in \mathbb{N}$ witness that $S_1(\emptyset)$ is not finite.