Model theory
Exercise ~~7~~

1. By induction on n, we show that if $p \in S_n(A)$, then p is realized in $\mathfrak{A}$.
$n=1$: Assumption
$n=m+1$: By the induction assumption there is $a \in \mathfrak{A}^m$ s.t. $\mathfrak{A} \models \exists v_1 \varphi(v_1, a, b)$ for all $\varphi(v_1, v_2, \dots, v_n, b) \in p$. By the assumption there is $c \in \mathfrak{A}$ s.t. $\mathfrak{A} \models \varphi(c, a, b)$ for all $\varphi(v_1, \dots, v_n, b) \in p$.

2. It is enough to show that if $\mathfrak{A} \leq \mathfrak{B} \models T_{acf_0}$ and $|\mathfrak{A}| < |\mathfrak{B}|$ and $p \in S_1(\mathfrak{A})$, p is realized in $\mathfrak{B}$. This follows from the observation that all elements in $\mathfrak{B} \setminus \mathfrak{A}$ satisfies the same formulas $\varphi(v, a)$, φ atomic, $a \in \mathfrak{A}^n$.

3. Let $\mathfrak{A} = \{a_i \mid i < k\}$ and f a partial elem. map $f: \mathfrak{A} \to \mathfrak{A}$ s.t. $|f| < k$. By induction on $i < k$ construct partial elem. $f_i: \mathfrak{A} \to \mathfrak{A}$ s.t.

(i) $f_0 = f$

(ii) $f_{i+1} \supseteq f_i$ and $a_i \in \mathrm{dom}(f_i) \cap \mathrm{rng}(f_i)$

(iii) if i is limit, then $f_i = \bigcup_{j<i} f_j$.

To extend f_i ~~so that~~ to g s.t.

$\mathrm{dom}(g) = \mathrm{dom}(f_i) \cup \{a_i\}$ just choose

$b \in \mathfrak{A}$ s.t. b realizes $\{ \varphi(v_1, f_i(a)) \mid$

$\varphi(v_1, a) \in t(a_i / \mathrm{dom}(f_i)) \}$ and

let $g = f_i \cup \{(a_i, b)\}$.

4. Let $(T, <)$ be as in the hint.

For each $f : \mu \to \kappa \setminus \{0\}$ let

$p_f = \{ f \restriction \alpha \cup \{(\alpha, 0)\} < v_1 < f \restriction \alpha \mid \alpha < \mu \}$

Then if $f \neq g$, $p_f \cup p_g$ is contradictory

but each p_f is consistent. Since there are

more than κ many of these no

model of size κ can realize them all.

5. Let $\mathfrak{A} = \bigcup_{\alpha < \kappa} \mathfrak{A}_\alpha$ where each $\mathfrak{A}_\alpha$

has power κ, for limit α, $\mathfrak{A}_\alpha = \bigcup_{\beta < \alpha} \mathfrak{A}_\beta$

and $\mathfrak{A}_{\alpha+1}$ realizes every $p \in S(\mathfrak{A}_\alpha)$.

Notice that if $A \subset \mathfrak{A}$ is of power $< \kappa$

then $A \subset \mathfrak{A}_\alpha$ for some $\alpha < \kappa$ since

κ is regular.