Model theory

Exercise 6

Hints

1. To find b, notice that if $x-y \geq 2^{n+2}$ then there is $y<z<x$ s.t. $z-y, y-x \geq 2^{n+1}$.

For the claim, suppose $X, \mathbb{N} \setminus X$ are infinite and $\varphi(x,y)$ and $c=(c_1,..,c_k)$ are such that φ is relational and $\varphi(x,c)$ defines X. Let n be the quantifier rank of φ. Then find $b \in X$ and $d \in \mathbb{N} \setminus X$ s.t. $\{(c_i, c_i) \mid 1 \leq i \leq k\} \cup \{(b,d)\}$ is n-good and go on showing that $\text{II} \uparrow EF_n((\mathfrak{A}, c, b), (\mathfrak{A}, c, d))$ (using n-good functions) and get a contradiction with 7.8.

2. Again suppose $X, \mathbb{N} \setminus X$ are infinite and $\varphi(x,c)$ defines X. For all $i<\omega$, let $\mathfrak{A}_i = \mathfrak{A}$ and D an ultrafilter on ω s.t. for all $n \in \omega$, $\{x \in \omega \mid x > n\} \in D$. Let $\mathfrak{B} = \prod_{i<\omega} \mathfrak{A}_i / D$ and by taking an isomorphic copy, we may assume

that $\mathfrak{A} \preceq \mathfrak{L}$. Now find $a \in g(\mathfrak{L}, c)$ and $b \in \mathfrak{L} \setminus g(\mathfrak{L}, c)$ and an automorphism f of $\mathfrak{L}$ s.t. $f(c) = c$ and $f(a) = b$. This is a contradiction.

3. Let $\mathfrak{B}$ be such that

$$dom(\mathfrak{B}) = \bigcup_{2 \le n < \omega} \{n\} \times \{0, \dots, n\}$$

and $f^{\mathfrak{B}}((n,i)) = (n, i+1)$ if $i < n$

and $f^{\mathfrak{B}}((n,n)) = (n, 0)$.

Let $\mathfrak{A} \succeq \mathfrak{B}$ be such that there is $a \in \mathfrak{A}$ s.t. for all $n < \omega$, $n \neq 0$, $(f^{\mathfrak{A}})^n(a) \neq a$ (use e.g. compactness).

4. II wins by playing all 3 games simultaniously.

5. $L = \{c_i \mid i < \omega\}$, $\mathfrak{A} = (\mathbb{N}, c_i^{\mathfrak{A}})$ where $c_i^{\mathfrak{A}} = i$ ja $\mathfrak{B} = (\mathbb{N}, c_i^{\mathfrak{B}})$ where $c_i^{\mathfrak{B}} = i + 1$. Notice that for all finite $L' \subseteq L$, $\mathfrak{A} \restriction L' \cong \mathfrak{B} \restriction L'$.