Model theory
~~#~~ Exercise 4
Hints

1. For $\mathfrak{B}, \mathfrak{C} \models T$, let $f : \mathfrak{A} \to \mathfrak{B}$ and $g : \mathfrak{A} \to \mathfrak{C}$ be as in the assumption. Apply AP to $g \circ f^{-1}$.

2. It is needed in the proof of 5.13 in the case when φ is a sentence.

3. Suppose $\mathfrak{A} \subseteq \mathfrak{B}, \mathfrak{C} \models T_{lo}$ are s.t. $\mathfrak{B} \cap \mathfrak{C} = \mathfrak{A}$. It is enough to find $\mathfrak{D} \models T_{lo}$ s.t. $\mathfrak{B}, \mathfrak{C} \subseteq \mathfrak{D}$.
For AP

Let $dom(\mathfrak{D}) = dom(\mathfrak{B}) \cup dom(\mathfrak{C})$ and extend $<^{\mathfrak{B}} \cup <^{\mathfrak{C}}$ to a linear ordering by defining: for $a \in \mathfrak{B} \setminus \mathfrak{A}$ and $b \in \mathfrak{C} \setminus \mathfrak{A}$, $a < b$ if there is $c \in \mathfrak{C}$ s.t. $a < c < b$ and o/w $b < a$.

Let $T = T_{lo} \cup \{\forall x \exists y (y < x), \forall x \exists y (x < y), \forall x \forall y \exists z (x < y \to x < z < y)\}$

4. AP as in 3 ($E^D = E^{\mathfrak{B}} \cup E^C$).

$T = T_{gr} \cup \{\varphi_n \mid n \in \omega\}$, where

$$\varphi_n = \forall x_0 \ldots \forall x_n \forall y_0 \ldots \forall y_n \exists z \Big(\big(\bigwedge_{\substack{i \leq n \\ j \leq n}} \neg x_i = y_j\big) \to \Big(\big(\bigwedge_{i \leq n} E(z, x_i)\big) \wedge \big(\bigwedge_{i \leq n} \neg E(z, y_i)\big) \wedge \big(\bigwedge_{i \leq n} \neg z = y_i\big)\Big)\Big).$$

5. Suppose $\psi(v_0)$ is q.f. and for all e.c. models $\mathfrak{A} \models T$ and $a \in \mathfrak{A}$,

$\mathfrak{A} \models \psi(a) \iff \mathfrak{A} \models \varphi(a)$.

Let $\mathfrak{A} \models T$ be e.c. Then for all $n \in \omega$ there is $a_n \in \mathfrak{A}$ s.t.

$\mathfrak{A} \models R(c_n, a_n) \wedge \bigwedge_{i < n} \neg R(c_i, a_n)$

Then $\mathfrak{A} \models \varphi(a_n) \wedge \psi(a_n)$.

Thus by compactness there are $\mathfrak{A} \leq \mathfrak{B} \models T$ and $b \in \mathfrak{B}$ s.t.

$\mathfrak{B} \models \varphi(b) \wedge \psi(b)$ and $\mathfrak{B} \models \neg R(c_n, b)$ for all $n \in \omega$. By Theorem 5.11, there is e.c. $\mathfrak{B} \subseteq \mathfrak{C} \models T$.

Then ~~x~~ $\mathfrak{C} \models \psi(b)$ since ψ is q.f.

but $\mathfrak{C} \not\models \varphi(b)$ because $\mathfrak{C} \models \neg R(c_n, b)$

for all $n \in \omega$ and $\mathfrak{C}$ is e.c., a

contradiction. $\square$