Model theory
Exercise 3

1-4: Just check the definitions

5. E.g. g is well-defined:

Suppose ~~[illegible]~~

$t^{\mathfrak{A}}(a) = u^{\mathfrak{A}}(b)$ where $t = t(x)$, $x = (x_1, .., x_n)$

$u = u(y)$, $y = (y_1, .., y_m)$, $a \in \mathrm{dom}(f)^n$

and $b \in \mathrm{dom}(f)^m$, w.a.l.g. x and

y are distinct. We need to show

that $t^{\mathfrak{B}}(f(a)) = u^{\mathfrak{B}}(f(b))$.

Look at the formula $\varphi = t = u$.

Then $\mathfrak{A} \models \varphi(a, b)$ and thus

by the definition of p.i.

$\mathfrak{B} \models \varphi(f(a), f(b))$ i.e. $t^{\mathfrak{B}}(f(a)) = u^{\mathfrak{B}}(f(b))$.

6 (i): Look at the theory

$T = Th(\mathfrak{A}, \mathfrak{A}) \cup \{\neg c = \underline{b} \mid b \in \mathfrak{A}\} \cup \{\varphi(c, \underline{a})\}$.

(ii): w.o.l.g. $|\mathfrak{A}| = \varkappa \geq |L_{\omega\omega}|$.

By induction on $i \leq k$ construct models

$\mathfrak{B}_i$, $i \leq \varkappa$, as follows:

$\mathfrak{D}_0 = \mathfrak{A}$ and for all $i < \kappa$ pick an enumeration $g_{ij}(v_0, a_{ij})$, $j < \kappa$, of all possible definitions in $\mathfrak{D}_i$.

Let $\pi : \kappa \to \kappa \times \kappa$ be a bijection s.t. if $\pi(i) = (\alpha, \beta)$, then $\alpha \leq i$.

$\mathfrak{A}_{i+1}$ is defined as follows:

Let $(\alpha, \beta) = \pi(i)$. If $g_{\alpha\beta}(\mathfrak{D}_i, a_{\alpha\beta})$ is finite, let $\mathfrak{D}_{i+1} = \mathfrak{D}_i$ o/w

Let $\mathfrak{D}_{i+1}$ be such that $g_{\alpha\beta}(\mathfrak{D}_{i+1}, a_{\alpha\beta}) \overset{(*)}{\not\subseteq} g_{\alpha\beta}(\mathfrak{D}_i, a_{\alpha\beta})$ (by (i))

At limits take union.

Then $\mathfrak{D}_\kappa$ is as wanted. $\square$

(*) and $\mathfrak{D}_i \preceq \mathfrak{D}_{i+1}$