Department of Mathematics and Statistics
Measure and Integral
Exercise 6
1.3-3.3.2017

Final exam is on March 15. See
https://wiki.helsinki.fi/display/mathstat/General+exams
for instructions.

1. Give an example of a sequence of continuous functions $f_i\colon [0,1] \to \mathbb{R}$ such that $\lim_{i\to\infty} f_i(x) = 0$ for all $x \in [0,1]$ but the sequence of Riemann integrals $\int_0^1 f_i(x)\,dx$ has no limit.

2. Find the limit
$$\lim_{k\to\infty} \int_0^{\pi} \frac{1}{\sqrt{x+\sin^k x}}\,dx.$$

3. Let $E_j \subset \mathbb{R}^n,\ j \in \mathbb{N}$, be a sequence of measurable disjoint sets and let $f\colon \mathbb{R}^n \to [0,+\infty]$ be measurable. Prove that
$$\int_{\cup_j E_j} f = \sum_{j=1}^{\infty} \int_{E_j} f.$$

4. ["Decreasing Monotone Convergence Theorem"]
Let $f_j\colon E \to \mathbb{R}$ be measurable functions such that $f_1 \geq f_2 \geq \cdots \geq 0$. Prove: If $\int_E f_1 < \infty$, then
$$\int_E \lim_{j\to\infty} f_j = \lim_{j\to\infty} \int_E f_j.$$

5. Let
$$f_n(x) = \frac{1+e^{-n|x|}}{1+x^2},\ x \in \mathbb{R},\ n \in \mathbb{N}.$$
Determine
$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\,dx.$$

6. Suppose that $E \subset \mathbb{R}^n$ is measurable and $f\colon E \to [0,+\infty]$ is a measurable function such that $\int_E f < \infty$. Define
$$E_j = \{x \in E\colon f(x) > j\},\ j \in \mathbb{N}.$$
Prove that
$$\lim_{j\to\infty} j \cdot m(E_j) = 0.$$