Measure and integral

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These are lecture notes of the course Measure and integral (Mitta ja integraali).

0 Some background

0.1 Basic operations on sets

Let X be an arbitrary set. The *power set* of X is the set of all subsets of X,

$$\mathcal{P}(X) = \{A\colon A \subset X\},$$

and any subset $\mathcal{F} \subset \mathcal{P}(X)$ is called a *family (or collection) of subsets of* X. The *union* of a family $\mathcal{F}$ is

$$\bigcup_{A\in\mathcal{F}} A = \{x \in X : x \in A \ \underline{\text{for some}}\ A \in \mathcal{F}\}$$

and the *intersection* (of $\mathcal{F}$) is

$$\bigcap_{A\in\mathcal{F}} A = \{x \in X : x \in A \ \underline{\text{for all}}\ A \in \mathcal{F}\}.$$

Let $\mathcal{A}$ be an index set (set of indices) and suppose that for every $\alpha \in \mathcal{A}$ there exists a unique subset $V_\alpha \subset X$. (In other words, $\alpha \mapsto V_\alpha$ is a mapping $\mathcal{A} \to \mathcal{P}(X)$.) Then the collection

$$\mathcal{F} = \{V_\alpha\colon \alpha \in \mathcal{A}\}$$

is an *indexed family of* X.

The union of an indexed family is

$$\bigcup_{\alpha\in\mathcal{A}} V_\alpha = \{x \in X : x \in V_\alpha \ \underline{\text{for some}}\ \alpha \in \mathcal{A}\}$$

and the intersection of an indexed family is

$$\bigcap_{\alpha\in\mathcal{A}} V_\alpha = \{x \in X : x \in V_\alpha \ \underline{\text{for all}}\ \alpha \in \mathcal{A}\}.$$

We denote also

$$\bigcup_{\alpha} V_\alpha \quad \text{and} \quad \bigcap_{\alpha} V_\alpha, \quad \text{if } \mathcal{A} \text{ is clear from the context.}$$

Example. 1. Let $\mathcal{F} \subset \mathcal{P}(X)$. We can interpret $\mathcal{F}$ as an indexed family by using $\mathcal{F}$ as the index set. That is, if $\alpha \in \mathcal{F}$ (thus α is a subset of X), we write $V_\alpha = \alpha$. Then $\mathcal{F} = \{V_\alpha\colon \alpha \in \mathcal{F}\}$.

2.
$$X = \bigcup_{x\in X} \{x\}, \qquad \{x\} = \text{ a singleton.}$$

If the index set is $\mathbb{N} = \{1, 2, 3, \dots\}$, we denote

$$\bigcup_{n\in\mathbb{N}} V_n \quad \text{or} \quad \bigcup_{n}^{\infty} V_n \quad \text{or} \quad \bigcup_{n} V_n,$$

and

$$\bigcap_{n\in\mathbb{N}} V_n \quad \text{or} \quad \bigcap_{n}^{\infty} V_n \quad \text{or} \quad \bigcap_{n} V_n.$$

Sequences (of sets) are denoted by (V_n), $(V_n)_{n=1}^{\infty}$, $(V_n)_{n\in\mathbb{N}}$, or $V_1, V_2, \ldots$.

The *difference* of sets $A, B \subset X$ is

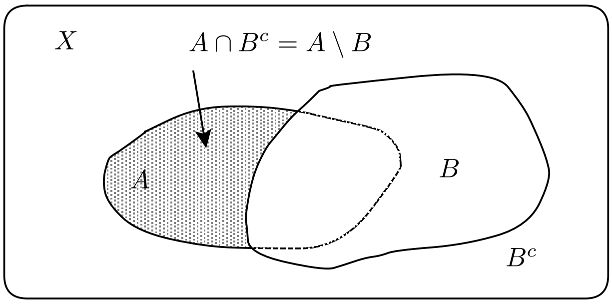
$$A \setminus B = \{x \in X \colon x \in A \text{ and } x \notin B\}.$$

The *complement* of a set $B \subset X$ (with respect to X) is

$$B^c = X \setminus B.$$

Remark.

$$A \setminus B = A \cap B^c.$$



Theorem 0.2. *Let* $\{V_\alpha \colon \alpha \in \mathcal{A}\}$ *be a family of* X*. Then the following* de Morgan's laws *hold:*

$$\Big(\bigcup_\alpha V_\alpha\Big)^c = \bigcap_\alpha V_\alpha^c \tag{0.3}$$

and

$$\Big(\bigcap_\alpha V_\alpha\Big)^c = \bigcup_\alpha V_\alpha^c. \tag{0.4}$$

Let $B \subset X$*. Then the following* distributive laws *for union and for intersection hold:*

$$B \cap \Big(\bigcup_\alpha V_\alpha\Big) = \bigcup_\alpha (B \cap V_\alpha) \tag{0.5}$$

and

$$B \cup \Big(\bigcap_\alpha V_\alpha\Big) = \bigcap_\alpha (B \cup V_\alpha). \tag{0.6}$$

Proof. (0.3):

$$x \in \Big(\bigcup_\alpha V_\alpha\Big)^c \iff x \notin \bigcup_\alpha V_\alpha \iff \forall \alpha \colon x \notin V_\alpha \iff \forall \alpha \colon x \in V_\alpha^c \iff x \in \bigcap_\alpha V_\alpha^c.$$

(0.4): Similarly.

(0.5):

$$\begin{aligned} x \in B \cap \Big(\bigcup_\alpha V_\alpha\Big) &\iff x \in B \text{ and } x \in \bigcup_\alpha V_\alpha \iff x \in B \text{ and } x \in V_\alpha \text{ for some } \alpha \in \mathcal{A} \\ &\iff x \in B \cap V_\alpha \text{ for some } \alpha \in \mathcal{A} \iff x \in \bigcup_\alpha (B \cap V_\alpha). \end{aligned}$$

(0.6): Similarly. ∎

The images and preimages of the union/intersection of a family.

Let X and Y be non-empty sets and $f\colon X \to Y$ a mapping.

The *image* of a set $A \subset X$ under the mapping f is

$$f(A) = \{f(x)\colon x \in A\}. \quad (\subset Y)$$

We usually abbreviate fA.

The *preimage* of a set $B \subset Y$ under the mapping f is

$$f^{-1}(B) = \{x \in X\colon f(x) \in B\}.$$

We also abbreviate $f^{-1}B$ and denote

$$f^{-1}(y) = f^{-1}(\{y\}),$$

if $y \in Y$. [Note: f need not have an inverse mapping.]

Theorem 0.7. *Let $f\colon X \to Y$ be a mapping and let $\{V_\alpha\colon \alpha \in \mathcal{A}\}$ be a family of X, and let $\{W_\beta\colon \beta \in \mathcal{B}\}$ be a family of Y. Then*

$$f\big(\bigcup_\alpha V_\alpha\big) = \bigcup_\alpha fV_\alpha \tag{0.8}$$

$$f^{-1}\big(\bigcup_\beta W_\beta\big) = \bigcup_\beta f^{-1}W_\beta \tag{0.9}$$

$$f^{-1}\big(\bigcap_\beta W_\beta\big) = \bigcap_\beta f^{-1}W_\beta. \tag{0.10}$$

Proof. (0.8):

$$\begin{aligned} y \in f\big(\bigcup_\alpha V_\alpha\big) &\iff y = f(x) \text{ and } x \in \bigcup_\alpha V_\alpha \iff y = f(x) \text{ and } x \in V_\alpha \text{ for some } \alpha \in \mathcal{A} \\ &\iff y \in fV_\alpha \text{ for some } \alpha \in \mathcal{A} \iff y \in \bigcup_\alpha fV_\alpha. \end{aligned}$$

(0.9) and (0.10): Similarly. □

Remark. It is always true that

$$f\big(\bigcap_\alpha V_\alpha\big) \subset \bigcap_\alpha fV_\alpha,$$

but the inclusion can be strict. The equality $f(\cap_\alpha V_\alpha) = \cap_\alpha fV_\alpha$ holds, for example, if f os an injection.

Countable and uncountable sets

Countability is a <u>very</u> important notion is measure theory!

Definition. A set A is *countable* if $A = \emptyset$ or there exists an injection $f\colon A \to \mathbb{N}$ ($\iff$ $\exists$ a surjection $g\colon \mathbb{N} \to A$).

A set A is *uncountable* if A is not countable.

Remark.

1. A countable $\iff$ A finite äärellinen (including $\emptyset$) or *countably infinite* (when there exists a bijection $f\colon A \to \mathbb{N}$).
2. A countable $\iff$ $A = \{x_n\colon n \in \mathbb{N}\}$ (repetition allowed, so that A can be finite).
3. A countable, $B \subset A$ $\Rightarrow$ B countable.

Theorem 0.11. *If the sets A_n are countable $\forall n \in \mathbb{N}$, then*

$$\bigcup_{n\in\mathbb{N}} A_n \text{ is countable.}$$

("countable union of countable sets is countable".)

Proof. We may assume that $A_n \neq \emptyset \ \forall \ n \in \mathbb{N}$. Since A_n is countable, we may write $A_n = \{x_m(n)\colon m \in \mathbb{N}\}$. Define a mapping

$$g\colon \mathbb{N}\times\mathbb{N} \to \cup_n A_n, \quad g(n,m) = x_m(n).$$

Then g is a surjection $\mathbb{N}\times\mathbb{N} \to \cup_n A_n$. Hence it suffices to find a surjection $h\colon \mathbb{N} \to \mathbb{N}\times\mathbb{N}$, because then

$$g\circ h\colon \mathbb{N} \to \bigcup_{n\in\mathbb{N}} A_n$$

is surjective and therefore $\cup_n A_n$ is countable. An example of a surjection $h\colon \mathbb{N} \to \mathbb{N}\times\mathbb{N}$ is:

$$\begin{array}{ccccccccccc}
\underset{=h(1)}{(1,1)} & & \underset{=h(3)}{(1,2)} & & \underset{=h(6)}{(1,3)} & & \underset{=h(10)}{(1,4)} & & \underset{=h(15)}{(1,5)} & \cdots \\
 & \nearrow & & \nearrow & & \nearrow & & \nearrow & & \\
\underset{=h(2)}{(2,1)} & & \underset{=h(5)}{(2,2)} & & \underset{=h(9)}{(2,3)} & & \underset{=h(14)}{(2,4)} & & & \\
 & \nearrow & & \nearrow & & \nearrow & & & & \\
\underset{=h(4)}{(3,1)} & & \underset{=h(8)}{(3,2)} & & \underset{=h(13)}{(3,3)} & & & & & \\
 & \nearrow & & \nearrow & & & & & & \\
\underset{=h(7)}{(4,1)} & & \underset{=h(12)}{(4,2)} & & & & & & & \\
 & \nearrow & & & & & & & & \\
\underset{=h(11)}{(5,1)} & & & & & & & & & \\
\vdots & & & & & & & & &
\end{array}$$

□

Corollary. The set of all rational numbers

$$\mathbb{Q} = \{\frac{m}{n} \mid n, m \in \mathbb{Z},\ n \neq 0\}$$

is countable. Reason: The set

$$A_k = \{\frac{m}{n} \mid n, m \in \mathbb{Z},\ n \neq 0,\ |m| \leq k, |n| \leq k\}$$

is finite (and hence countable) $\forall k \in \mathbb{N}$. Theorem 0.11 $\Rightarrow$ $\mathbb{Q} = \cup_{k\in\mathbb{N}} A_k$ countable. □

Example. (Uncountable set). The interval $[0,1]$ (and hence $\mathbb{R}$) is uncountable.

Idea: $x \in [0,1] \Rightarrow$ x has a decimal expansion

$$x = 0, a_1 a_2 a_3 \ldots,$$

where $a_j \in \{0, 1, 2, \ldots, 9\}$.

Contrapositive: $[0,1]$ is countable, so $[0,1] = \{x_n : n \in \mathbb{N}\}$. Points x_n have decimal expansions

$$\begin{aligned} x_1 &= 0, a_1^{(1)} a_2^{(1)} a_3^{(1)} \ldots \\ x_2 &= 0, a_1^{(2)} a_2^{(2)} a_3^{(2)} \ldots \\ x_3 &= 0, a_1^{(3)} a_2^{(3)} a_3^{(3)} \ldots \\ &\vdots \\ x_n &= 0, a_1^{(n)} a_2^{(n)} a_3^{(n)} \ldots a_n^{(n)} \ldots \\ &\vdots \end{aligned}$$

On the "diagonal" there is a sequence $a_1^{(1)}, a_2^{(2)}, a_3^{(3)}, \ldots, a_n^{(n)}, \ldots$, where $a_n^{(n)}$ is the nth decimal of x_n. Let $x \in [0,1]$ be defined by $x = 0, b_1 b_2 b_3 \ldots$, where

$$b_n = \begin{cases} a_n^{(n)} + 2, & \text{if } a_n^{(n)} \in \{0, 1, 2, \ldots, 7\}, \\ a_n^{(n)} - 2, & \text{if } a_n^{(n)} \in \{8, 9\}. \end{cases} \tag{0.12}$$

The nth decimal of x satisfies $|b_n - a_n^n| = 2$ $\forall n \in \mathbb{N}$, and therefore $x \neq x_n$ $\forall n \in \mathbb{N}$. This is a contradiction, because $[0,1] = \{x_n : n \in \mathbb{N}\}$. Hence $[0,1]$ is uncountable.
[Note: A decimal expansion need not be unique: for instance, $0,5999\ldots = 0,6000\ldots$. However, this makes no harm, because in (0.12) $b_n = a_n^{(n)} \pm 2$.]

Infinite sums.

Let $\mathcal{A} \neq \emptyset$ be an arbitrary index set and $a_\alpha \geq 0$ $\forall \alpha \in \mathcal{A}$. Question: What does the sum

$$\sum_{\alpha\in\mathcal{A}} a_\alpha$$

mean?

Define

$$\sum_{\alpha\in\mathcal{A}} a_\alpha = \sup\{\sum_{\alpha\in\mathcal{A}_0} a_\alpha \mid \mathcal{A}_0 \subset \mathcal{A} \text{ finite}\}.$$

We will return to this a bit later.

0.13 Euclidean space $\mathbb{R}^n$

$$\mathbb{R}^n = \overbrace{\mathbb{R} \times \cdots \times \mathbb{R}}^{n \text{ times}} \quad \text{Cartesian product}$$

The elements are called *points* or *vectors*.

$$x \in \mathbb{R}^n \iff x = (x_1, \ldots, x_n),\ x_j \in \mathbb{R},\ j = 1, \ldots, n.$$

Algebraic structure.
The *sum* of points $x, y \in \mathbb{R}^n$ is

$$x + y = (x_1 + y_1, \ldots, x_n + y_x) \in \mathbb{R}^n.$$

The *product* of a real number $\lambda \in \mathbb{R}$ and a point $x \in \mathbb{R}^n$ is

$$\lambda x = (\lambda x_1, \ldots, \lambda x_n) \in \mathbb{R}^n.$$

Zero vector

$$0 = \bar{0} = (0, \ldots, 0).$$

The *inverse element (point)* of $x \in \mathbb{R}^n$ is

$$-x = (-1)x = (-x_1, \ldots, -x_n).$$

The *difference* of $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ is

$$x - y = x + (-y).$$

In $\mathbb{R}^n$ the addition and multiplication by a real number satisfy the axioms of a *vector space*, for example

$$x + y = y + x, \quad x + 0 = 0 + x = x,$$
$$\lambda(x + y) = \lambda x + \lambda y, \quad (\lambda + \mu)x = \lambda x + \mu x \quad \text{etc}$$
$$\forall x, y \in \mathbb{R}^n, \lambda, \mu \in \mathbb{R}.$$

The *inner product* of $x, y \in \mathbb{R}^n$ is

$$x \cdot y = \sum_{i=1}^{n} x_i y_i \in \mathbb{R}.$$

Denote

$$|x| = \sqrt{x \cdot x} = \left(\sum_{i=1}^{n} x_i x_i \right)^{1/2} \qquad \textit{norm} \text{ of } x.$$

The Euclidean distance in $\mathbb{R}^n$.
The *distance* between $x, y \in \mathbb{R}^n$ is

$$|x - y| = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}.$$

Often we write $d(x,y) = |x-y|$. Then d is a *metric* in $\mathbb{R}^n$, i.e. the mapping $d\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies the axioms of a metric:

$$\begin{aligned}
&d(x,y) \geq 0 \quad \forall x,y \in \mathbb{R}^n \\
&d(x,y) = 0 \iff x = y \\
&d(x,y) = d(y,x) \quad \forall x,y \in \mathbb{R}^n \\
&d(x,y) \leq d(x,z) + d(z,y) \quad \forall x,y,z \in \mathbb{R}^n \quad \text{(triangle inequality, } \triangle\text{-ie).}
\end{aligned}$$

Open sets and closed sets in $\mathbb{R}^n$.

The Euclidean metric d determines open and closed sets of $\mathbb{R}^n$ (and hence the topology of $\mathbb{R}^n$) as follows:

Let $x \in \mathbb{R}^n$ and $r > 0$. The set

$$B(x,r) = \{y \in \mathbb{R}^n \colon |y-x| < r\}$$

is an *open ball* with the center x and radius r and

$$S(x,r) = \{y \in \mathbb{R}^n \colon |y-x| = r\}$$

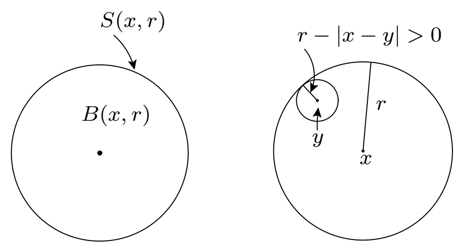
is the *sphere* (centered at x and with radius r. Similarly,

$$\bar{B}(x,r) = \{y \in \mathbb{R}^n \colon |y-x| \leq r\}$$

is a *closed ball* (centered at x with radius r).

A set $V \subset \mathbb{R}^n$ is *open* if $\forall x \in V\ \exists\ r = r(x) > 0$ such that $B(x,r) \subset V$.

A set $V \subset \mathbb{R}^n$ is *closed* is $\mathbb{R}^n \setminus V$ is open.



Example. 1. $B(x,r)$ is open $\forall x \in \mathbb{R}^n, r > 0$ ($\triangle$-ie, see the picture above).

2. A closed ball $\bar{B}(x,r)$ is a closed set.

3. $\mathbb{R}^n$ and $\emptyset$ are both open and closed.

4. A half open interval, e.g. $[0,1)$, is neither open nor closed.

Remark. The *closure* of a set $A \subset \mathbb{R}^n$ is

$$\overline{A} = \{x \in \mathbb{R}^n \colon x \in A \text{ or } x \text{ is an accumulation (or a cluster) point of } A\}.$$

Recall that $x \in \mathbb{R}^n$ is an accumulation point of $A \subset \mathbb{R}^n$ if $\forall r > 0\ B(x,r) \cap (A \setminus \{x\}) \neq \emptyset$. In $\mathbb{R}^n$ it holds that $\bar{B}(x,r) = \overline{B(x,r)}$.

Remark. If (X,d) is a *metric space*, i.e. $d\colon X \times X \to \mathbb{R}$ satisfies the axioms of a metric, we can define open and closed sets of X by using th emetric d as in the case of $\mathbb{R}^n$ by replacing $|y-x|$ with the metric $d(x,y)$.

The following result holds in general:

Theorem 0.14.

$$V_\alpha \subset \mathbb{R}^n \textit{ open } \forall \alpha \in \mathcal{A} \textit{ (arbitrary index set)} \Rightarrow \bigcup_{\alpha \in \mathcal{A}} V_\alpha \textit{ open;} \tag{0.15}$$

$$V_\alpha \subset \mathbb{R}^n \textit{ closed } \forall \alpha \in \mathcal{A} \Rightarrow \bigcap_{\alpha \in \mathcal{A}} V_\alpha \textit{ closed;} \tag{0.16}$$

$$V_1, \ldots, V_k \subset \mathbb{R}^n \textit{ open} \Rightarrow \bigcap_{j=1}^{k} V_j \textit{ open;} \tag{0.17}$$

$$V_1, \ldots, V_k \subset \mathbb{R}^n \textit{ closed} \Rightarrow \bigcup_{j=1}^{k} V_j \textit{ closed.} \tag{0.18}$$

Proof. (0.15):

$$x \in \bigcup_{\alpha \in \mathcal{A}} V_\alpha \Rightarrow \exists \alpha_0 \in \mathcal{A} \text{ s.t. } x \in V_{\alpha_0},$$
$$V_{\alpha_0} \text{ open } \Rightarrow \exists \text{ open ball } B(x,r) \subset V_{\alpha_0} \subset \bigcup_{\alpha \in \mathcal{A}} V_\alpha.$$

(0.16):

$$\begin{aligned} V_\alpha \quad &\text{closed } \forall\ \alpha \ \Rightarrow\ V_\alpha^c \quad \text{open } \forall\ \alpha \\ \overset{(0.15)}{\Longrightarrow} \quad &\bigcup_\alpha V_\alpha^c \overset{\text{de Morgan}}{=} \Big(\bigcap_\alpha V_\alpha\Big)^c \quad \text{open} \\ \Rightarrow &\bigcap_\alpha V_\alpha \quad \text{closed.} \end{aligned}$$

(0.17) and (0.18): (Exerc.). □

Remark.

$$V_j \text{ open } \forall j \in \mathbb{N} \not\Rightarrow \bigcap_{j=1}^{\infty} V_j \text{ open,}$$
$$V_j \text{ closed } \forall j \in \mathbb{N} \not\Rightarrow \bigcup_{j=1}^{\infty} V_j \text{ closed.} \quad \text{(Exerc.)}$$

1 Lebesgue measure in $\mathbb{R}^n$

1.1 Introduction

A geometric starting point: If $I = [a, b] \subset \mathbb{R}$ is a bounded interval, its length is

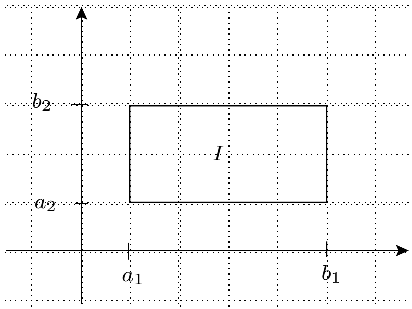
$$\ell(I) = b - a.$$

(Similarly if I is an open or half open interval.)

A set $I \subset \mathbb{R}^n$ is an *n-interval* if it is of the form

$$I = I_1 \times \cdots \times I_n,$$

where each $I_j \subset \mathbb{R}$ is an interval (either open, closed, or half open).



An n-interval I is an *open* (respectively *closed*) *n-interval* if each I_j is open (resp. closed).

Let I_j has the end points $a_j, b_j;\ a_j < b_j$. Then the *geometric measure* of I is

$$\ell(I) = (b_1 - a_1)(b_2 - a_2)\cdots(b_n - a_n) = \prod_{j=1}^{n}(b_j - a_j)$$

($n = 1$ length, $n = 2$ area, $n = 3$ volume). Define $\ell(\emptyset) = 0$.

Our goal would be to define a "measure" as a mapping

$$m_n\colon \mathcal{P}(\mathbb{R}^n) \to [0, +\infty],$$

such that it satisfies the conditions:

(1) $m_n(E)$ is defined $\forall\ E \subset \mathbb{R}^n$ and $m_n(E) \geq 0$.

(2) If I is an n-interval, then $m_n(I) = \ell(I)$.

(3) If (E_k) is a sequence of *disjoint* subsets of $\mathbb{R}^n$ (i.e. $E_j \cap E_k = \emptyset$ if $j \neq k$), then

$$m_n\big(\cup_{k=1}^{\infty} E_k\big) = \sum_{k=1}^{\infty} m_n(E_k) \quad \textit{countably additivity.}$$

(4) m_n is *translation invariant*, i.e.

$$m_n(E + x) = m_n(E),$$

where $E \subset \mathbb{R}^n,\ x \in \mathbb{R}^n$, and $E + x = \{y + x \mid y \in E\}$.

It turns out that there exists *no such mapping* that would satisfy all the conditions (1) – (4) simultaneously. In the case of the (n-dimensional) Lebesgue measure m_n we drop the condition (1). Hence

$$m_n\colon \operatorname{Leb}\mathbb{R}^n \to [0, +\infty],$$

will be a mapping that satisfies the conditions (2), (3) and (4), where

$$\operatorname{Leb}\mathbb{R}^n \subsetneq \mathcal{P}(\mathbb{R}^n)$$

is the family of *Lebesgue measurable sets*. The family $\operatorname{Leb}\mathbb{R}^n$ contains, for instance, all open and closed subsets of $\mathbb{R}^n$.

1.2 The Lebesgue outer measure in $\mathbb{R}^n$

Convention.

$$\begin{gathered} a+\infty=\infty+a=\infty, \qquad a\neq-\infty \\ a-\infty=-\infty+a=-\infty, \qquad a\neq\infty \\ \infty-\infty, \quad -\infty+\infty \quad \text{not defined} \\ -(\infty)=-\infty, \qquad -(-\infty)=\infty \end{gathered}$$

$$\infty\cdot a=a\cdot\infty=\begin{cases}\infty, & a>0\\ -\infty, & a<0\\ 0, & a=0\end{cases} \qquad \text{Note! } 0\cdot\infty=0$$

$$(-\infty)a=a(-\infty)=\begin{cases}-\infty, & a>0\\ +\infty, & a<0\\ 0, & a=0\end{cases}$$

$$\begin{gathered} \infty\cdot\infty=(-\infty)(-\infty)=\infty \\ (-\infty)\infty=\infty(-\infty)=-\infty \end{gathered}$$

$$\frac{a}{0}=\begin{cases}\infty, & a>0\\ -\infty, & a<0\\ \text{not defined}\ , & a=0\end{cases}$$

$$\frac{a}{\infty}=\frac{a}{-\infty}=0, \quad a\in\mathbb{R}$$

$$\frac{\pm\infty}{\pm\infty} \quad \text{not defined}$$

Recall: If $(a_j)_{j\in\mathbb{N}}$ is a sequence such that $a_j\geq 0\ \forall\ j$, then either

$$\sum_{j=1}^{\infty}a_j=\lim_{k\to\infty}\sum_{j=1}^{k}a_j\in\mathbb{R} \quad \text{or} \quad \sum_{j=1}^{\infty}a_j=+\infty.$$

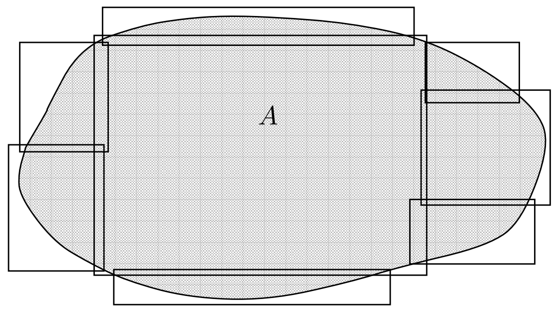
Reason: partial sums $\sum_{j=1}^{k}a_j$ form an increasing sequence.

Let $A\subset\mathbb{R}^n$. Consider *countable open covers of* A (possibly finite)

$$\mathcal{F}=\{I_1,I_2,\ldots\},$$

where each $I_k\subset\mathbb{R}^n$ is a bounded open n-interval (or $\emptyset$) and

$$A\subset\bigcup_{k=1}^{\infty}I_k.$$



Then we say that $\mathcal{F}$ is a *Lebesgue cover* of A. We form a series

$$S(\mathcal{F}) = \sum_{k=1}^{\infty} \ell(I_k), \quad 0 < S(\mathcal{F}) \leq +\infty.$$

Definition. The *n-dimensional (Lebesgue) outer measure of A* is

$$\boxed{m_n^*(A) = \inf \{S(\mathcal{F})\colon\ \mathcal{F} \text{ is a Lebesgue cover of } A\}.}$$

(Later we will prove that closed n-intervals would work as well.)

Remark. 1. Denote $J_k = \{x \in \mathbb{R}^n\colon |x_j| < k\ \forall j\}$ (open n-interval). Clearly

$$\mathbb{R}^n = \bigcup_{k=1}^{\infty} J_k,$$

and therefore always there exist open covers $\cup_{k=1}^{\infty} I_k \supset A$ (and hence inf exists).

2. $I_k \subset \mathbb{R}^n$ open n-interval $\Rightarrow\ 0 \leq \ell(I_k) < \infty\ \Rightarrow$ the sum is well-defined and

$$0 \leq \sum_{k=1}^{\infty} \ell(I_k) \leq +\infty.$$

3. The outer measure $m_n(A)$ depends (of course) on the dimension n. If n is clear from the context, we abbreviate $m^*(A) = m_n^*(A)$.

4. It follows directly from the definition that $\forall \varepsilon > 0$ there exists a Lebesgue cover $\mathcal{F}$ of A (usually depending on ε) such that

$$S(\mathcal{F}) \leq m^*(A) + \varepsilon.$$

(We allow $m^*(A) = +\infty$.) Note that it is usually not possible to find a Lebesgue cover $\mathcal{F}$ of A for which $m_n^*(A) = S(\mathcal{F})$.

5. Thus $A \mapsto m^*(A)$ is a mapping $\mathcal{P}(\mathbb{R}^n) \to [0, \infty]$, in particular, m^* is defined in the whole $\mathcal{P}(\mathbb{R}^n)$.

Example. 1. Let $n = 2$ and let $A = \{(x, 0)\colon a \leq x \leq b\} \subset \mathbb{R}^2$ (a line segment in the plane).
Claim: $m_2^*(A) = 0$.
Proof: Let $\varepsilon > 0$ and $I_\varepsilon =]a - \varepsilon, b + \varepsilon[\times] - \varepsilon, \varepsilon[\subset \mathbb{R}^2$ an open 2-interval.

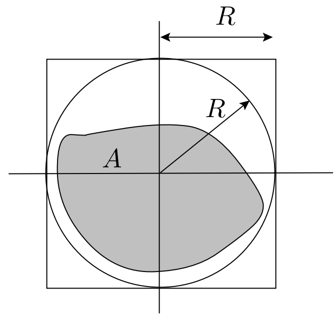
$$A \subset I_\varepsilon \Rightarrow 0 \leq m_2^*(A) \leq \ell(I_\varepsilon) = 2\varepsilon(b - a + 2\varepsilon) \xrightarrow{\varepsilon \to 0} 0,$$

hence $m_2^*(A) = 0$.

2. Let $n = 1$. Consider the set of rational numbers $\mathbb{Q} \subset \mathbb{R}$.
   Claim: $m_1^*(\mathbb{Q}) = 0$.
   Proof Since $\mathbb{Q}$ is countable, we may write $\mathbb{Q} = \{q_j : j \in \mathbb{N}\}$. Let $\varepsilon > 0$ be arbitrary. For each $j \in \mathbb{N}$ let
$$I_j = \left]q_j - \frac{\varepsilon}{2^{j+1}}, q_j + \frac{\varepsilon}{2^{j+1}}\right[\subset \mathbb{R}$$
   be an open interval. Its length is $\ell(I_j) = 2\varepsilon/2^{j+1} = \varepsilon/2^j$.
$$q_j \in I_j \quad \forall j \in \mathbb{N} \Rightarrow \mathbb{Q} \subset \bigcup_j I_j \Rightarrow$$
$$0 \le m_1^*(\mathbb{Q}) \le \sum_{j=1}^{\infty} \ell(I_j) = \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon \sum_{j=1}^{\infty} \frac{1}{2^j} = \varepsilon \xrightarrow{\varepsilon \to 0} 0,$$
   hence $m_1^*(\mathbb{Q}) = 0$.

3. Similarly, $A \subset \mathbb{R}^n$ countable $\Rightarrow$ $m_n^*(A) = 0$.

4. Let $A \subset \mathbb{R}^n$ be a *bounded* set, that is $\exists R > 0$ such that $A \subset B(0, R)$. Then $A \subset I$, where
$$I = \overbrace{]-R, R[\times \cdots \times]-R, R[}^{n \text{ times}} \quad \text{open } n\text{-interval.}$$



   We get an estimate
$$m^*(A) \le \ell(I) = (2R)^n.$$

Basic properties of the (Lebesgue) outer measure.

Theorem 1.3. *(1)* $m_n^*(\emptyset) = 0$;

(2) "monotonicity": $A \subset B \Rightarrow m_n^*(A) \le m_n^*(B)$;

(3) "subadditivity": $A_1, A_2, \ldots \subset \mathbb{R}^n \Rightarrow$
$$m_n^*\Big(\bigcup_{j=1}^{\infty} A_j\Big) \le \sum_{j=1}^{\infty} m_n^*(A_j).$$

Remark. (3) holds also for finite unions $\cup_{j=1}^k (A_j)$ (choose $A_{k+1} = \cdots = \emptyset$).

Proof. (1): Clear.

(2): Let $\mathcal{F}$ be a Lebesgue cover of B.

$$A \subset B \quad \Rightarrow \quad \mathcal{F} \text{ is also a Lebesgue cover of } A \quad \overset{\text{definition}}{\Longrightarrow} \quad m_n^*(A) \leq S(\mathcal{F}).$$

Take the inf over all Lebesgue covers of $B \quad \Rightarrow \quad m_n^*(A) \leq m_n^*(B)$.

(3): Denote $A = \cup_j A_j$. Let $\varepsilon > 0$. For each j choose a Lebesgue cover $\mathcal{F}_j = \{I_{j1}, I_{j2} \ldots\}$ of A_j such that

$$S(\mathcal{F}_j) \leq m_n^*(A_j) + \varepsilon/2^j.$$

Now $\mathcal{F} = \bigcup_j \mathcal{F}_j = \{I_{jk} \colon j \in \mathbb{N}, k \in \mathbb{N}\}$ is a Lebesgue cover of A, hence (by definition)

$$m_n^*(A) \leq S(\mathcal{F}) = \sum_{j=1}^{\infty} S(\mathcal{F}_j) \leq \sum_{j=1}^{\infty} m_n^*(A_j) + \sum_{j=1}^{\infty} \varepsilon/2^j = \sum_{j=1}^{\infty} m_n^*(A_j) + \varepsilon.$$

Letting $\varepsilon \to 0$ we get the claim.

□

Remark. Above we need some facts on "summing" (more precisely, why $S(\mathcal{F}) = \sum_{j=1}^{\infty} S(\mathcal{F}_j)$)? See Lemma 1.7 and 1.8 below.

Theorem 1.4. *Let $A \subset \mathbb{R}^n$. Then*

$$m_n^*(A + x) = m_n^*(A) \tag{1.5}$$

for all $x \in \mathbb{R}^n$, where $A + x = \{y + x \colon y \in A\}$;

$$m_n^*(tA) = t^n m_n^*(A), \tag{1.6}$$

whenever $t > 0$ and $tA = \{ty \colon y \in A\}$.

Proof (Exerc.) □

On summing. Let I be an (index) set and $a_i \geq 0 \; \forall \; i \in I$. If $J \subset I$ is finite, we denote

$$S_J = \sum_{i \in J} a_i \,, \qquad S_\emptyset = 0.$$

Definition.

$$\sum_{i \in I} a_i = \sup\{S_J \colon J \subset I \text{ finite}\}.$$

Lemma 1.7.

$$\sum_{i \in \mathbb{N}} a_i = \lim_{n \to \infty} \sum_{i=1}^{n} a_i.$$

That is, this "new" definition coincide with the usual one (for countable sums).

Proof Denote $J_n = \{1, \ldots, n\}$, $\quad S = \sum_{i \in \mathbb{N}} a_i \quad (= \sup\{S_J \colon J \subset \mathbb{N} \text{ finite}\})$.

$$\left(S_{J_n}\right) \text{ increasing sequence} \quad \Rightarrow \quad \exists \lim_{n \to \infty} S_{J_n} = S'$$
$$S_{J_n} \leq S \quad \Rightarrow \quad S' \leq S.$$

On the other hand,

$$\begin{aligned} J\subset\mathbb{N}\text{ finite} \quad &\Rightarrow \quad \exists\, n\in\mathbb{N}\ \text{ s.t. }\ J\subset J_n \\ &\Rightarrow \quad S_J\le S_{J_n}\le S' \\ &\Rightarrow \quad S\le S' \quad (\text{taking sup over } \forall\ J). \end{aligned}$$

□

Next both I and J are arbitrary index sets (i.e. they may be uncountable). (In addition, we abbreviate $a_{ij}=a_{(i,j)}$.)

Lemma 1.8.

$$\sum_{(i,j)\in I\times J} a_{ij} = \sum_{i\in I}\sum_{j\in J} a_{ij} = \sum_{j\in J}\sum_{i\in I} a_{ij}.$$

Proof Denote by S_{vas} the sum on the left hand side, by S_{kes} the sum in the middle, and by S_{oik} the sum on the right hand side.

(a): If $\mathcal{A}\subset I\times J$ is finite, then $\exists$ finite $I'\subset I,\ J'\subset J$ s.t. $\mathcal{A}\subset I'\times J'$

$$\begin{aligned} \Rightarrow \quad S_{\mathcal{A}} \le S_{I'\times J'} &\overset{(*)}{=} \sum_{i\in I'}\sum_{j\in J'} a_{ij} \le \sum_{i\in I'}\sum_{j\in J} a_{ij} \le S_{\text{kes}} \\ \Rightarrow \quad S_{\text{vas}} &\le S_{\text{kes}} \quad (\text{taking sup over } \forall\ \mathcal{A}). \end{aligned}$$

[(*): there is only finitely many terms in $S_{I'\times J'}$, so the order of summing does not matter.]

(b): Let $I'\subset I$ be finite and $J_i'\subset J$ be finite $\forall\ i\in I'$. Denote

$$\mathcal{A}=\{(i,j)\colon i\in I', j\in J_i'\}.$$

Then

$$S_{\text{vas}} \ge S_{\mathcal{A}} = \sum_{i\in I'}\sum_{j\in J_i'} a_{ij}.$$

Take ($\forall\ i\in I'$) the sup over finite $J_i'\subset J$

$$S_{\text{vas}} \ge \sum_{i\in I'}\sum_{j\in J} a_{ij}$$

$$\text{sup over finite } I'\subset I \quad \Rightarrow \quad S_{\text{vas}} \ge S_{\text{kes}}.$$

Similarly, $S_{\text{vas}} = S_{\text{oik}}$. □

Corollary 1.9.

$$\sum_{(i,j)\in\mathbb{N}\times\mathbb{N}} a_{ij} = \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} a_{ij} = \sum_{j\in\mathbb{N}}\sum_{i\in\mathbb{N}} a_{ij}\,.$$

Remark. The subadditivity <u>does not</u> (in general) hold in the form

$$m_n^*\Big(\bigcup_{i\in I} A_i\Big) \le \sum_{i\in I} m_n^*(A_i), \tag{1.10}$$

where $A_i\subset\mathbb{R}^n,\ i\in I$, and I is an *uncountable* index set. <u>Reason:</u>

$$\mathbb{R}^n = \bigcup_{x\in\mathbb{R}^n}\{x\}, \quad m_n^*(\{x\})=0\ \forall x\in\mathbb{R}^n.$$

If (1.10) would hold, then

$$0 \le m_n^*(\mathbb{R}^n) = m_n^*\Big(\bigcup_{x\in\mathbb{R}^n}\{x\}\Big) \overset{(1.10)}{\le} \sum_{x\in\mathbb{R}^n} m_n^*(\{x\}) = 0.$$

On the other hand, we will prove later that $m_n^*(\mathbb{R}^n) = +\infty$. This is a contradiction, so (1.10) does not hold!

1.11 (Lebesgue)measurable sets

We will define the (Lebesgue) measurable sets of $\mathbb{R}^n$, denoted by $\operatorname{Leb}\mathbb{R}^n$, by using so-called *Carathéodory's condition.*

Recall the subadditivity (Theorem 1.3 (3)): $A, B \subset \mathbb{R}^n \quad \Rightarrow$

$$m^*(A \cup B) \le m^*(A) + m^*(B).$$

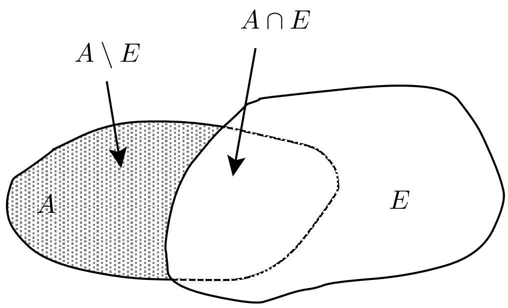
Later we will prove that $\exists\, A, B \subset \mathbb{R}^n$ s.t. $A \cap B = \emptyset$, but

$$m^*(A \cup B) < m^*(A) + m^*(B).$$

In other words, the Lebesgue outer measure m^* is not countable additive. We want to get rid of this unsatisfactory behaviour and therefore we "throw away" certain sets.

Let $E \subset \mathbb{R}^n$ be given and let $A \subset \mathbb{R}^n$ be a "test set":

$$A = (A \cap E) \cup (A \setminus E) \quad \text{disjoint union}$$
$$m^* \text{ subadditive } \Rightarrow m^*(A) \le m^*(A \cap E) + m^*(A \setminus E).$$



Definition. (Carathéodory's condition, 1914.) A set $E \subset \mathbb{R}^n$ is *(Lebesgue) measurable* if

$$m^*(A) = m^*(A \cap E) + m^*(\underbrace{A \setminus E}_{=A\cap E^c}) \quad \underline{\text{for all}} \quad A \subset \mathbb{R}^n.$$

Remark. $E \subset \mathbb{R}^n$ measurable $\iff$

$$m^*(A) \ge m^*(A \cap E) + m^*(A \setminus E) \quad \text{for all } A \subset \mathbb{R}^n, \text{ with } m^*(A) < \infty.$$

<u>Reason:</u> $\boxed{\le}$ follows from the subadditivity and $\boxed{\ge}$ holds always if $m^*(A) = +\infty$.

Definition. If $E \subset \mathbb{R}^n$ is measurable, we denote

$$m(E) = m^*(E) \quad \text{or } m_n(E) \text{ if needed.}$$

$m(E)$ is the *(n-dimensional Lebesgue) measure* of E.

<u>We write</u>

$$\operatorname{Leb}\mathbb{R}^n = \{E \subset \mathbb{R}^n \colon E \text{ Lebesgue measurable}\} \quad \subset \mathcal{P}(\mathbb{R}^n).$$

Hence

$$m = m^*|\operatorname{Leb}\mathbb{R}^n : \operatorname{Leb}\mathbb{R}^n \to [0,\infty], \quad \text{restriction of the outer measure.}$$

Later we will show that

$$\operatorname{Leb}\mathbb{R}^n \subsetneq \mathcal{P}(\mathbb{R}^n).$$

Theorem 1.12.

$$m^*(E) = 0 \quad \Rightarrow \quad E \textit{ measurable.}$$

Proof. Let $A \subset \mathbb{R}^n$ be an arbitrary test set.

$$A \cap E \subset E \overset{\text{monotonicity}}{\Longrightarrow} m^*(A \cap E) = 0$$

$$A \supset A \setminus E \overset{\text{monotonocity}}{\Longrightarrow} m^*(A) \geq m^*(A \setminus E) = \underbrace{m^*(A \cap E)}_{=0} + m^*(A \setminus E)$$

$$\Rightarrow \quad E \text{ measurable.}$$

□

Theorem 1.13.

$$E \quad \textit{measurable} \quad \Longleftrightarrow \quad E^c \quad \textit{measurable.}$$

Proof. It s enough to show $\boxed{\Rightarrow}$: Let E be measurable and $A \subset \mathbb{R}^n$. Then

$$\begin{aligned} m^*(A) &= m^*(A \cap E) + m^*(A \cap E^c) \\ &= m^*\big(A \cap (E^c)^c\big) + m^*(A \cap E^c) \\ &\Rightarrow \quad E^c \quad \text{measurable.} \end{aligned}$$

□

Example.

$$E \subset \mathbb{R}^n \quad \text{countable} \overset{\text{Ex. 3}}{\Longrightarrow} m^*(E) = 0$$

$$\overset{\text{Thm. 1.12}}{\Longrightarrow} E \quad \text{measurable} \overset{\text{Thm. 1.13}}{\Longrightarrow} E^c \quad \text{measurable.}$$

Special cases:

$$\emptyset \in \operatorname{Leb}\mathbb{R}, \quad \mathbb{R} \in \operatorname{Leb}\mathbb{R},$$

$$\text{rational numbers } \mathbb{Q} \in \operatorname{Leb}\mathbb{R}, \text{ irrational numbers } \mathbb{R} \setminus \mathbb{Q} \in \operatorname{Leb}\mathbb{R}.$$

Let $E_1,\ E_2, \ldots$ be measurable. We will prove that

$$\bigcup_{i=1}^{\infty} E_i \quad \text{and} \quad \bigcap_{i=1}^{\infty} E_i \quad \text{are measurable.}$$

To prove these statements we need some auxiliary lemmata. First the case of a finite union/intersection:

Lemma 1.14. $E_1, \ldots, E_k$ *measurable* $\Rightarrow \bigcup_{i=1}^k E_i$ *and* $\bigcap_{i=1}^k E_i$ *measurable.*

Proof. (a) union:

$$\bigcup_{i=1}^{k} E_i = \left(\bigcup_{i=1}^{k-1} E_i\right) \cup E_k$$

$\Rightarrow$ we may assume $k = 2$.

Suppose E_1 and E_2 are measurable. Let $A \subset \mathbb{R}^n$ be a test set.

$$\left.\begin{array}{c} E_1 \quad \text{measurable} \quad \Rightarrow \\ m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c) \\ \\ E_2 \quad \text{measurable, with test set } A \cap E_1^c \quad \Rightarrow \\ m^*(A \cap E_1^c) = m^*(A \cap E_1^c \cap E_2) + m^*(A \cap E_1^c \cap E_2^c) \end{array}\right\} \Longrightarrow$$

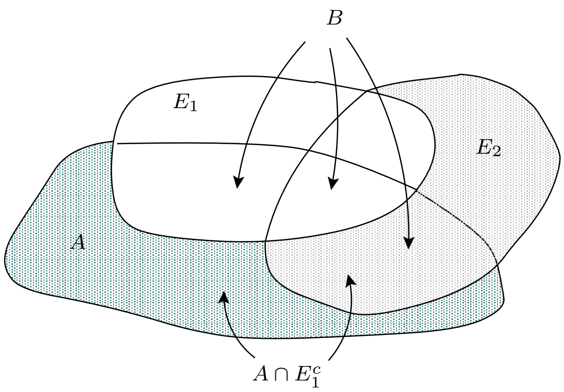
$$m^*(A) = \underbrace{m^*(A \cap E_1) + m^*(A \cap E_1^c \cap E_2)}_{(\text{subadd. } \Rightarrow) \ \geq \ m^*(B)} + m^*(A \cap E_1^c \cap E_2^c),$$

where

$$\begin{aligned} B &= (A \cap E_1) \cup (A \cap E_1^c \cap E_2) = A \cap \big(E_1 \cup (E_1^c \cap E_2)\big) = A \cap \big(E_1 \cup (E_2 \setminus E_1)\big) \\ &= A \cap (E_1 \cup E_2). \end{aligned}$$

Hence

$$\begin{aligned} m^*(A) &\geq m^*(B) + m^*(A \cap E_1^c \cap E_2^c) \\ &= m^*\big(A \cap (E_1 \cup E_2)\big) + m^*\big(A \cap (E_1 \cup E_2)^c\big) \\ &\Rightarrow \quad E_1 \cup E_2 \quad \text{measurable.} \end{aligned}$$



(b) intersection: de Morgan, Theorem 1.13 ("measurability of the complement") and part (a) $\Rightarrow$

$$\bigcap_{i=1}^{k} E_i = \left(\bigcup_{i=1}^{k} E_i^c\right)^c \quad \text{measurable.}$$

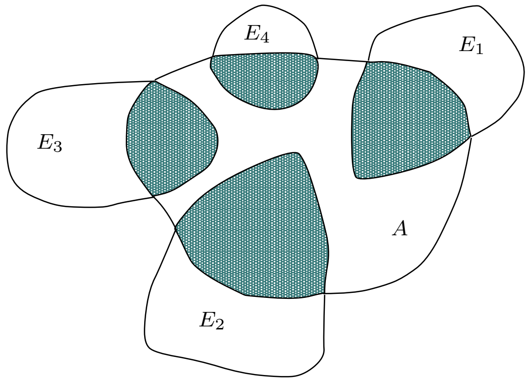
□

Theorem 1.15. *E_1, E_2 measurable $\Rightarrow$ $E_1 \setminus E_2$ measurable.*

Proof. $E_1 \setminus E_2 = E_1 \cap E_2^c$. □

Lemma 1.16. *Let $E_1, \dots, E_k$ be disjoint and measurable, and let $A \subset \mathbb{R}^n$ be an arbitrary set. Then*

$$m^*\big(A \cap (\bigcup_{i=1}^{k} E_i)\big) = \sum_{i=1}^{k} m^*(A \cap E_i).$$



Proof. (a) The case $k = 2$: E_1 measurable, $A \cap (E_1 \cup E_2) = B$ as the test set $\Rightarrow$

$$m^*(B) = m^*(\underbrace{B \cap E_1}_{=A\cap E_1}) + m^*(\underbrace{B \setminus E_1}_{=A\cap E_2})$$

$$= m^*(A \cap E_1) + m^*(A \cap E_2) \quad \text{i.e. the claim.}$$

(b) general case: By induction: Suppose that the claim holds for $2 \le k \le p$, that is

$$\left.\begin{array}{c} E_1, \dots, E_p \text{ measurable} \\ E_i \cap E_j = \emptyset,\ i \neq j \\ A \subset \mathbb{R}^n \end{array}\right\} \Rightarrow\ m^*\big(A \cap (\bigcup_{i=1}^{p} E_i)\big) = \sum_{i=1}^{p} m^*(A \cap E_i).$$

Thus we get (for $k = p + 1$)

$$\left.\begin{array}{l} A \cap \big(\bigcup_{i=1}^{p+1} E_i\big) = A \cap \big((\bigcup_{i=1}^{p} E_i) \cup E_{p+1}\big) \\ \\ \bigcup_{i=1}^{p} E_i,\ E_{p+1} \ \text{ disjoint and measurable} \end{array}\right\} \Longrightarrow$$

$$m^*\big(A \cap (\bigcup_{i=1}^{p+1} E_i)\big) \overset{k=2}{=} m^*\big(A \cap (\bigcup_{i=1}^{p} E_i)\big) + m^*(A \cap E_{p+1})$$

$$\overset{k=p}{=} \sum_{i=1}^{p} m^*(A \cap E_i) + m^*(A \cap E_{p+1})$$

$$= \sum_{i=1}^{p+1} m^*(A \cap E_i).$$

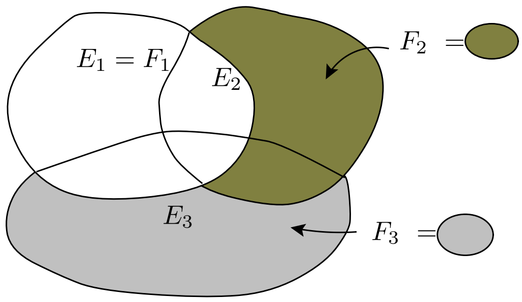
□

Lemma 1.17. *Let $E = \bigcup_{i=1}^{\infty} E_i$, where the sets E_i are measurable. Then there exist* disjoint *and measurable sets $F_i \subset E_i$ s.t.*

$$E = \bigcup_{i=1}^{\infty} F_i.$$

Proof. Choose

$$\begin{aligned} &F_1 = E_1, \qquad \text{[measurable]}\\ &F_2 = E_2 \setminus E_1, \qquad \text{[measurable (Thm. 1.15)]}\\ &\quad \vdots\\ &F_k = E_k \setminus \bigcup_{i=1}^{k-1} E_i, \qquad \text{[measurable (Thm. 1.15 and L. 1.14)]}\\ &\quad \vdots \end{aligned}$$



Then clearly

$$F_i \subset E_i \quad \forall\, i, \quad E = \bigcup_{i=1}^{\infty} F_i \quad \text{and} \quad F_i \cap F_j = \emptyset \;\forall\; i \neq j.$$

□

The main result of Lebesgue measurable sets

Theorem 1.18. *Let $E_1, E_2, \ldots$ be a sequence (possibly finite) of measurable sets. Then the sets*

$$\bigcup_i E_i \quad \text{and} \quad \bigcap_i E_i$$

are measurable. If, in addition, the sets E_i are disjoint, then

(1.19)
$$m\Big(\bigcup_i E_i\Big) = \sum_i m(E_i). \qquad \text{("countably additivity")}$$

Proof. Denote

$$\begin{aligned} &S = \bigcup_i E_i \overset{1.17}{=} \bigcup_i F_i, \quad F_i \text{ measurable and disjoint,}\\ &S_k = \bigcup_i^k F_i, \quad S_k \subset S. \end{aligned}$$

L. 1.14 (measurability of finite unions) $\Rightarrow$ S_k measurable. Let A be a test set. Then

$$\begin{aligned} m^*(A) &= m^*(A \cap S_k) + m^*(A \setminus S_k)\\ &\overset{\text{monot.}}{\geq} m^*(A \cap S_k) + m^*(A \setminus S)\\ &\overset{1.16}{=} \sum_{i=1}^{k} m^*(A \cap F_i) + m^*(A \setminus S) \quad \forall k \in \mathbb{N}. \end{aligned}$$

Letting $k \to \infty$ we get

$$
\begin{aligned}
(1.20) \qquad m^*(A) &\geq \sum_{i=1}^{\infty} m^*(A \cap F_i) + m^*(A \setminus S) \\
&\overset{\text{subadd.}}{\geq} m^*\big(\cup_{i=1}^{\infty}(A \cap F_i)\big) + m^*(A \setminus S) \\
&= m^*(A \cap S) + m^*(A \setminus S) \\
&\Rightarrow S = \bigcup_i E_i \quad \text{measurable.}
\end{aligned}
$$

Inequality (1.20), in the case $A = S$, and the subadditivity $\Rightarrow$

$$\sum_i^{\infty} m(F_i) \overset{\text{subadd.}}{\geq} m(S) \overset{(1.20)}{\geq} \sum_{i=1}^{\infty} m^*(\overbrace{S \cap F_i}^{=F_i}) + \overbrace{m^*(S \setminus S)}^{=0} = \sum_{i=1}^{\infty} m(F_i).$$

If E_i are disjoint, we may choose $F_i = E_i$, and therefore (1.19) holds.

The first part of the proof and Thm. 1.13 imply that $\bigcap_i E_i = \big(\bigcup_i E_i^c\big)^c$ is measurable. □

Example. Let $A \subset \mathbb{R}^2$ s.t.

$$(1.21) \qquad m^*\big(A \cap B(x,r)\big) \leq |x| r^3 \quad \forall x \in \mathbb{R}^2,\ \forall r > 0.$$

<u>Claim:</u> $m(A) = 0$

<u>Proof.</u> (a) Suppose first that A is bounded, so $A \subset Q = [-a,a] \times [-a,a]$ (closed square) for some a. Let $n \in \mathbb{N}$. Devide Q into closed (sub-)squares Q_j, with side length $= 2a/n,\ j = 1, \dots, n^2$. Let x_j be the center of Q_j. Then

$$
\begin{gathered}
|x_j| \leq 2a \quad \text{and} \quad Q_j \subset B(x_j, 2a/n) \qquad \text{(rough estimates)} \\
\Rightarrow\ m^*(A \cap Q_j) \overset{\text{monot.}}{\leq} m^*\big(A \cap B(x_j, 2a/n)\big) \overset{(1.21)}{\leq} |x_j|(2a/n)^3 \leq (2a)^4 n^{-3}. \\
A = \bigcup_{j=1}^{n^2} (A \cap Q_j) \overset{\text{subadd.}}{\Longrightarrow} \\
m^*(A) = m^*\big(\bigcup_{j=1}^{n^2}(A \cap Q_j)\big) \leq \sum_{j=1}^{n^2} m^*(A \cap Q_j) \\
\leq n^2 (2a)^4 n^{-3} = (2a)^4 n^{-1} \quad \forall\ n \\
\overset{n\to\infty}{\Longrightarrow} m^*(A) = 0 \ \Rightarrow\ m(A) = 0.
\end{gathered}
$$

(b) General case:

$$A = \bigcup_{j \in \mathbb{N}} A_j, \text{ where } A_j = A \cap B(0,j) \text{ bounded.}$$

$$
\begin{gathered}
A_j \subset A \ \Rightarrow\ A_j \text{ satisfies the assumption (1.21)} \overset{(a)}{\Rightarrow} m(A_j) = 0\ \forall j \\
\overset{\text{subadd.}}{\Longrightarrow} m(A) = 0.
\end{gathered}
$$

1.22 Examples of measurable sets

So far we know that:

$$m^*(A) = 0 \quad \Rightarrow \; A \text{ and } A^c \text{ measurable.}$$

Now we will prove that, for example, open sets and closed sets are measurable.
First:

$$I \subset \mathbb{R}^n \quad n\text{-interval (open, closed, etc.)} \quad \Rightarrow \quad I \text{ is measurable and } m(I) = \ell(I).$$

We use (Riemann) integration:

Let $I = I_1 \times \cdots \times I_n \subset \mathbb{R}^n$ n-interval, where $I_j \subset \mathbb{R}$ is an interval, with end points $a_j < b_j,\ j = 1, \ldots, n$. Let $\chi_I \colon \mathbb{R}^n \to \{0, 1\}$ (the characteristic function of I)

$$\chi_I(x) = \begin{cases} 1, & x \in I \\ 0, & x \notin I\,. \end{cases}$$

Choose an n-interval $Q \supset I$ and (Riemann) integrate

$$\int_Q \chi_I = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} 1\, dx_1 \cdots dx_n = (b_1 - a_1) \cdots (b_n - a_n) = \ell(I).$$

Lemma 1.23. *Let I and $I_1, \ldots, I_k$ be n-intervals s.t. $I \subset \bigcup_{j=1}^k I_j$. Then $\ell(I) \le \sum_{j=1}^k \ell(I_j)$. If, furthermore, the intersections $I_i \cap I_j,\ i \ne j$, do not have interior points (i.e. no $I_i \cap I_j,\ i \ne j$, contains an open ball) and $I = \bigcup_{j=1}^k I_j$, then $\ell(I) = \sum_{j=1}^k \ell(I_j)$.*

Proof. Define $\chi, \chi_j \colon \mathbb{R}^n \to \{0, 1\}$,

$$\chi(x) = \begin{cases} 1, & x \in I \\ 0, & x \notin I \end{cases} \qquad \text{and} \qquad \chi_j(x) = \begin{cases} 1, & x \in I_j \\ 0, & x \notin I_j. \end{cases}$$

Then it follows from the assumption $I \subset \bigcup_{j=1}^k I_j$ that $\chi(x) \le \sum_{j=1}^k \chi_j(x)\ \forall\, x \in \mathbb{R}^n$. Choose an n-interval Q that contains all the n-intervals mentioned above and (Riemann) integrate over Q

$$\ell(I) = \int_Q \chi \le \int_Q \big(\sum_j \chi_j\big) = \sum_j \int_Q \chi_j = \sum_j \ell(I_j).$$

If the n-intervals I_j do not have common interior points, then $\chi(x) = \sum_{j=1}^k \chi_j(x)$ except possible on the boundaries of n-intervals that do not contribute to the integrals. □

Lemma 1.24. *If I is an n-interval, then*

$$m^*(I) = \ell(I).$$

Proof. (a): $\forall \varepsilon > 0\ \exists$ an open n-interval $J \supset I$ s.t. $\ell(J) < \ell(I) + \varepsilon$.

$$\{J\} \quad \text{Leb. cover of } I \;\Rightarrow\; m^*(I) \le \ell(I) + \varepsilon$$
$$\varepsilon > 0 \text{ arbitr. } \Rightarrow\; m^*(I) \le \ell(I).$$

(b): Suppose first that I is closed. Let $\mathcal{F}$ be a Lebesgue cover of I. Since I is closed and bounded, I is compact. So $\exists$ a finite subcover $\mathcal{F}_0 = \{I_1, \dots, I_k\} \subset \mathcal{F}$. Lemma 1.23 $\Rightarrow$

$$\begin{gathered} \ell(I) \leq S(\mathcal{F}_0) \leq S(\mathcal{F}) \\ \inf \text{ over } \forall \mathcal{F} \;\Rightarrow\; \ell(I) \leq m^*(I). \end{gathered}$$

Hence: $\ell(I) = m^*(I)$ if I is closed. Suppose then that I need not be closed. Let $\varepsilon > 0$. Now $\exists$ a closed n-interval $I_c \subset I$ s.t. $\ell(I_c) > \ell(I) - \varepsilon$. Thus

$$\begin{gathered} m^*(I) \overset{\text{monot.}}{\geq} m^*(I_c) = \ell(I_c) > \ell(I) - \varepsilon \\ \varepsilon > 0 \text{ arbitr.} \;\Rightarrow\; m^*(I) \geq \ell(I). \end{gathered}$$

□

Remark. The above holds also for *degenerate* n-intervals $I = I_1 \times \cdots \times I_n \subset \mathbb{R}^n$, where at least one I_j is a singleton. Then $\ell(I) \overset{\text{def.}}{=} 0 = m_n^*(I)$.

Let $A \subset \mathbb{R}^n$, $\varepsilon > 0$ and let $J_1, J_2, \ldots \subset \mathbb{R}^n$ be arbitrary n-intervals s.t. $A \subset \bigcup_{i=1}^\infty J_i$. For each i $\exists$ *open* n-interval $I_i \supset J_i$ s.t. $\ell(I_i) < \ell(J_i) + \varepsilon/2^i$. Now $\{I_1, I_2 \ldots\}$ is a Lebesgue cover of A, and therefore $m^*(A) \leq \sum_{i=1}^\infty \ell(I_i) \leq \sum_{i=1}^\infty \ell(J_i) + \varepsilon$. (Recall a geometric series.) It follows that

$$m^*(A) = \inf\Big\{\sum_{i=1}^\infty \ell(J_i) \colon A \subset \bigcup_{i=1}^\infty J_i,\; J_i \text{ arbitrary } n\text{-interval}\Big\}.$$

Theorem 1.25. *If I is an n-interval, then I is measurable and*

$$m(I) = \ell(I).$$

Proof. L. 1.24 $\Rightarrow$ it suffices to prove that I is measurable. Let $A \subset \mathbb{R}^n$ be a test set. Claim:

$$m^*(A) \geq m^*(A \cap I) + m^*(A \setminus I).$$

Let $\varepsilon > 0$. Then $\exists$ a Lebesgue cover of A by open n-intervals $\mathcal{F} = \{I_1, I_2, \ldots\}$ s.t.

$$S(\mathcal{F}) \leq m^*(A) + \varepsilon.$$

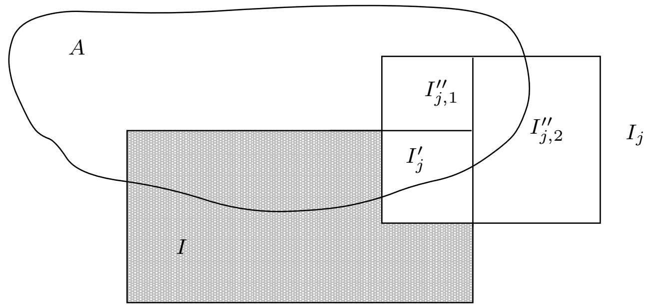
$$\left.\begin{array}{c} I = \Delta_1 \times \cdots \times \Delta_n \\[2ex] I_j =]a_1, b_1[\times \cdots \times]a_n, b_n[\end{array}\right\} \Rightarrow$$

$$I_j \cap I = \big(]a_1, b_1[\cap \Delta_1\big) \times \cdots \times \big(]a_n, b_n[\cap \Delta_n\big) = \begin{cases} n\text{-interval } I_j' \\ \emptyset. \end{cases}$$

$I_j \setminus I$ is not necessarily an n-interval but

$$I_j \setminus I = \bigcup_k I''_{j,k}$$

is a finite union of n-intervals s.t. the intersections $I_j' \cap I''_{j,k}$ and $I''_{j,k} \cap I''_{j,i}$, $k \neq i$, do not have interior points.



Lemma 1.23 and 1.24 $\Rightarrow$

$$\ell(I_j) \overset{1.23}{=} \ell(I'_j) + \sum_k \ell(I''_{j,k}) \overset{1.24}{=} m^*(I'_j) + \sum_k m^*(I''_{j,k}).$$

Taking the sum over j $\Rightarrow$

$$\begin{aligned} m^*(A) + \varepsilon \geq & S(\mathcal{F}) = \sum_j \ell(I_j) = \sum_j m^*(I'_j) + \sum_j \sum_k m^*(I''_{j,k}) \\ & \overset{\text{subadd.}}{\geq} m^*\Big(\underbrace{\bigcup_j I'_j}_{\supset A \cap I}\Big) + m^*\Big(\underbrace{\bigcup_{j,k} I''_{j,k}}_{\supset A \setminus I}\Big) \\ & \overset{\text{monot.}}{\geq} m^*(A \cap I) + m^*(A \setminus I). \end{aligned}$$

Letting $\varepsilon \to 0$ $\Rightarrow$ $m^*(A) \geq m^*(A \cap I) + m^*(A \setminus I)$. □

Theorem 1.26. *(Lindelöf's theorem) Let $A \subset \mathbb{R}^n$ be an arbitrary set and*

$$\bigcup_{\alpha \in \mathcal{A}} V_\alpha \supset A,$$

where the sets $V_\alpha \subset \mathbb{R}^n$, $\alpha \in \mathcal{A}$ are open. Then there exists a countable *sub-cover*

$$\bigcup_{j \in \mathbb{N}} V_{\alpha_j} \supset A.$$

Proof. Exerc. □

Theorem 1.27. *Open subsets and closed subsets of $\mathbb{R}^n$ are measurable.*

Proof. (a) Let A be open. If $x \in A$, $\exists$ an open n-interval $I(x)$ s.t. $x \in I(x) \subset A$ ($\exists$ an open ball $B(x, r_x) \subset A$ and it contains an open n-interval).

$$\{I(x) \colon x \in A\} \quad \text{is an open cover of } A.$$

Lindelöf $\Rightarrow$ $\exists$ countable sub-cover $\{I(x_j) \colon j \in \mathbb{N}\}$

$$\begin{aligned} &\Rightarrow A = \bigcup_{j \in \mathbb{N}} I(x_j) \quad \text{is a countable union of measurable sets} \\ &\Rightarrow A \quad \text{is measurable.} \end{aligned}$$

(b) If A is closed, its complement A^c is open and hence measurable $\Rightarrow$ $A = (A^c)^c$ is measurable. □

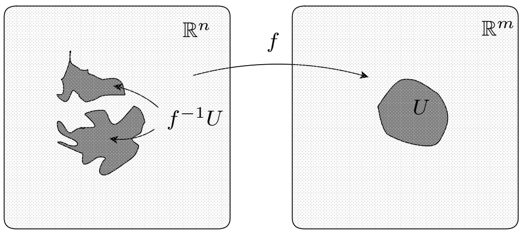
Example. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be continuous. Claim: $f\mathbb{R}^2$ is measurable.

Proof.

$$\mathbb{R}^2 = \bigcup_{j\in\mathbb{N}} A_j, \quad \text{where } A_j = \bar{B}(0,j) \text{ si compact}$$
$$f \text{ continuous } \Rightarrow fA_j \text{ compact}$$
$$\Rightarrow fA_j \text{ closed } \Rightarrow fA_j \text{ measurable}$$
$$f\mathbb{R}^2 = \bigcup_{j\in\mathbb{N}} fA_j \Rightarrow f\mathbb{R}^2 \text{ measurable.}$$

□

Recall: Let $n, m \geq 1$. A mapping $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is continuous $\iff$ $f^{-1}U \subset \mathbb{R}^n$ is open $\forall$ open $U \subset \mathbb{R}^m$.



If $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is continuous and $C \subset \mathbb{R}^n$ is compact, then $fC \subset \mathbb{R}^m$ is compact. Reason:

$$fC \subset \bigcup_{i\in I} U_i \quad \text{open cover}$$
$$\left.\begin{aligned} \Rightarrow C \subset \textstyle\bigcup_{i\in I} f^{-1}U_i \quad \text{open cover} \\ C \text{ compact} \end{aligned}\right\} \Rightarrow \exists \text{ finite sub-cover}$$
$$C \subset \bigcup_{j=1}^{k} f^{-1}U_{i_j} \Rightarrow fC \subset \bigcup_{j=1}^{k} U_{i_j}.$$

More general measurable sets, σ-algebras.

$$\mathcal{F}_\sigma \text{ sets } \bigcup_{i\in\mathbb{N}} F_i, \quad F_i \text{ closed} \quad (\text{e.g. } \mathbb{Q},\ [a,b),\ (a,b])$$
$$\mathcal{G}_\delta \text{ sets } \bigcap_{i\in\mathbb{N}} G_i, \quad G_i \text{ open} \quad (\text{e.g. } \mathbb{R}\setminus\mathbb{Q},\ [a,b),\ (a,b])$$
$$\mathcal{F}_{\sigma\delta} \text{ sets } \bigcap_{i\in\mathbb{N}} A_j,\ A_j \in \mathcal{F}_\sigma$$
$$\mathcal{G}_{\delta\sigma} \text{ sets } \bigcup_{i\in\mathbb{N}} B_j,\ B_j \in \mathcal{G}_\delta$$

etc.

Definition. Let X be an arbitrary set. A family $\Gamma \subset \mathcal{P}(X)$ is a σ-*algebra* ("sigma-algebra") of X if

(a) $\emptyset \in \Gamma$;

(b) $A \in \Gamma \;\Rightarrow\; X \setminus A \in \Gamma$;

(c) $A_i \in \Gamma,\; i \in \mathbb{N} \;\Rightarrow\; \bigcup_{i=1}^{\infty} A_i \in \Gamma$.

Remark. (1) If Γ is a σ-algebra and $A_i \in \Gamma,\; i \in \mathbb{N}$, then also $\bigcap_i A_i \in \Gamma$ since

$$\bigcap_i A_i = \bigcap_i (A_i^c)^c = (\bigcup_{i=1} A_i^c)^c \in \Gamma.$$

(2) We have proved: The family of Lebesgue measurable sets $\mathrm{Leb}\,\mathbb{R}^n$ is a σ-algebra of $\mathbb{R}^n$ (Theorems 1.12, 1.13, 1.18).

(3) $\mathcal{P}(X)$ is the largest σ-algebra of X; $\{\emptyset, X\}$ is the smallest σ-algebra of X; $A \subset X$ (fixed) $\Rightarrow$ $\{\emptyset, X, A, A^c\}$ is a σ-algebra of X.

Definition. The family of *Borel sets* $\mathrm{Bor}\,\mathbb{R}^n$ is the smallest σ-algebra of $\mathbb{R}^n$ that contains all closed sets.

Existence: Denote

$$\boxed{\mathcal{B} = \bigcap \{\Gamma \colon \Gamma \text{ is a } \sigma\text{-algebra of } \mathbb{R}^n,\; \Gamma \text{ contains closed sets}\}.}$$

(For instance $\Gamma = \mathcal{P}(\mathbb{R}^n)$ is a σ-algebra of $\mathbb{R}^n$ that contains all closed sets.)
$\mathcal{B}$ is a σ-algebra since:

(a) $\emptyset \in \mathcal{B}$;

(b) $A \in \mathcal{B} \;\Rightarrow\; A^c \in \Gamma\ \forall \Gamma \;\Rightarrow\; A^c \in \mathcal{B}$;

(c) $A_i \in \mathcal{B} \;\Rightarrow\; \bigcup_{i \in \mathbb{N}} A_i \in \Gamma\ \forall \Gamma \;\Rightarrow\; \bigcup_{i \in \mathbb{N}} A_i \in \mathcal{B}$.

The construction $\Rightarrow$ $\mathcal{B}$ is the smallest σ-algebra of $\mathbb{R}^n$ that contains closed sets, and so

$$\boxed{\mathrm{Bor}\,\mathbb{R}^n = \mathcal{B}\,.}$$

Open sets, closed sets, $\mathcal{F}_\sigma$ sets, $\mathcal{G}_\delta$ sets, etc. are Borel sets.

Theorem 1.28. *Every Borel sets is measurable.*

Proof. The family of measurable sets $\mathrm{Leb}\,\mathbb{R}^n$ is a σ-algebra and contains closed sets, and therefore

$$\mathrm{Bor}\,\mathbb{R}^n \subset \mathrm{Leb}\,\mathbb{R}^n.$$

□

1.29 General measure theory

Definition. Let Γ be a σ-algebra in X. A function $\mu\colon \Gamma \to [0,+\infty]$ is a *measure* in X if

(i) $\mu(\emptyset) = 0$;

(ii) $A_i \in \Gamma,\ i \in \mathbb{N},$ disjoint $\Rightarrow\ \mu\big(\bigcup_{i=1}^{\infty} A_i\big) = \sum_{i\in\mathbb{N}} \mu(A_i)$. "countably additivity"

The triple (X, Γ, μ) is a *measure space*.

Remark. 1. A measure μ is also *monotonic*:

$$A, B \in \Gamma,\ A \subset B\ \Rightarrow\ 0 \le \mu(A) \le \mu(B).$$

Reason: $A,\ B\setminus A \in \Gamma$ disjoint, $B = A \cup (B \setminus A)$

$$\Rightarrow\ \mu(B) = \mu(A) + \underbrace{\mu(B\setminus A)}_{\ge 0} \ge \mu(A).$$

2. $A, B \in \Gamma,\ A \subset B,\ \mu(A) < \infty\ \Rightarrow\ \mu(B\setminus A) = \mu(B) - \mu(A)$.

3. A measure μ is a *probability measure* if $\mu(X) = 1$.

Example. (1) n-dimensional Lebesgue measure

$$m_n\colon \operatorname{Leb}\mathbb{R}^n \to [0,+\infty]$$

is a measure.
Reason: $\operatorname{Leb}\mathbb{R}^n$ is a σ-algebra in $\mathbb{R}^n$ and m is countably additive.

(2) Let $X \neq \emptyset$ be an arbitrary set. Fix $x \in X$ and define for all $A \subset X$

$$\mu(A) = \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{if } x \notin A. \end{cases}$$

Then $\mu\colon \mathcal{P}(X) \to [0,+\infty]$ is a probability measure (so-called *Dirac measure* at the point $x \in X$).
Reason: (a) $\mathcal{P}(X)$ is σ-algebra.
(b) Let $A_j \subset X,\ j \in \mathbb{N}$, be disjoint. Then

$$\mu\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \sum_{j=1}^{\infty} \mu(A_j)$$

since

$$\begin{cases} x \notin \bigcup_{j=1}^{\infty} A_j\ \Rightarrow\ \text{both sides } = 0 \\[2ex] x \in \bigcup_{j=1}^{\infty} A_j \overset{\text{disjoint}}{\Longrightarrow} \exists \text{ exactly one } j_0 \in \mathbb{N} \text{ s.t. } x \in A_{j_0}\ \Rightarrow\ \text{both sides } = 1. \end{cases}$$

(3) $\mu\colon \mathcal{P}(X) \to [0,+\infty],\ \mu(A) = 0\ \forall A \subset X$, is a measure.

(4) Let $a_j \geq 0,\ j \in \mathbb{N}$, s.t. $\sum_{j=1}^{\infty} a_j = 1$. Define for all $A \subset \mathbb{N}$

$$\mu(A) = \sum_{j \in A} a_j.$$

Then $\mu\colon \mathcal{P}(\mathbb{N}) \to [0,1]$ is a probability measure.

Definition. Let X be an arbitrary set. A mapping $\mu^*\colon \mathcal{P}(X) \to [0,+\infty]$ is an *outer measure* in X if

(1) $\mu^*(\emptyset) = 0$;

(2) $A \subset B \ \Rightarrow\ \mu^*(A) \leq \mu^*(B)$;

(3) $A_j \subset X,\ j \in \mathbb{N} \ \Rightarrow\ \mu^*\big(\bigcup_{j=1}^{\infty} A_j\big) \leq \sum_{j=1}^{\infty} \mu^*(A_j)$.

Furthermore, a set $E \subset X$ is (μ^*-)*measurable*, if (Carathéodory's criterion)

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E) \tag{1.30}$$

holds $\forall A \subset X$.
Denote

$$\mathcal{M}_{\mu^*}(X) = \{E \subset X \colon E\ \mu^*\text{-measurable}\}$$

of $\mathcal{M}(X)$ is μ^* is clear from the context.

Remark. $\mathcal{M}(X) \subset \mathcal{P}(X)$ is a σ-algebra in X and the restriction

$$\mu^*|\mathcal{M}(X)\colon \mathcal{M}(X) \to [0,+\infty]$$

is a measure. Proof as in the case of Lebesgue measure.

1.31 Convergence of measures

Let $X \neq \emptyset,\ \Gamma \subset \mathcal{P}(X)$ a σ-algebra, and $\mu\colon \Gamma \to [0,+\infty]$ a measure.

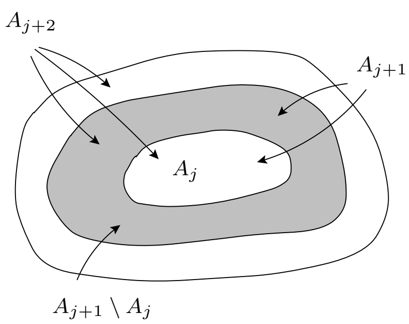
Theorem 1.32. *Let $A_j \in \Gamma,\ j = 1,\dots,$ be an increasing sequence (i.e. $A_1 \subset A_2 \subset \cdots \subset X$ (μ-)measurable). Then*

$$\mu\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \lim_{j\to\infty} \mu(A_j).$$

Note: $A_j \in \Gamma\ \forall j \in \mathbb{N} \ \Rightarrow\ \bigcup_{j=1}^{\infty} A_j \in \Gamma$.

Proof.

$$\bigcup_{j=1}^{\infty} A_j = \bigcup_{j=1}^{\infty} (\underbrace{A_j \setminus A_{j-1}}_{\text{disjoint, measurable}}), \quad A_0 = \emptyset \ \ \text{(a convention)}$$



μ countably additive $\Rightarrow$

$$\begin{aligned}\mu\Big(\bigcup_{j=1}^{\infty} A_j\Big) &= \sum_{j=1}^{\infty} \mu(A_j \setminus A_{j-1}) \\ &= \lim_{k\to\infty} \sum_{j=1}^{k} \mu(A_j \setminus A_{j-1}) \\ &= \lim_{k\to\infty} \mu\underbrace{\Big(\bigcup_{j=1}^{k} (A_j \setminus A_{j-1})\Big)}_{=A_k} \\ &= \lim_{k\to\infty} \mu(A_k).\end{aligned}$$

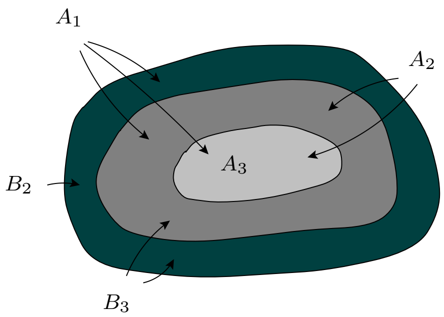
□

Theorem 1.33. *Let $A_j \in \Gamma$, $j = 1, \ldots,$ be a* <u>*decreasing*</u> *sequence (i.e. $X \supset A_1 \supset A_2 \supset \cdots$ (μ-)measurable). If, in addition, $\mu(A_k) < \infty$ for some $k \in \mathbb{N}$, then*

$$\mu\Big(\bigcap_{j=1}^{\infty} A_j\Big) = \lim_{j\to\infty} \mu(A_j).$$

<u>Note:</u> Γ σ-alg. $\Rightarrow$ $\bigcap_{j=1}^{\infty} A_j \in \Gamma$.

Proof. We may assume that $\mu(A_1) < \infty$. Denote $\bigcap_{j=1}^{\infty} A_j = A$ and $B_j = A_1 \setminus A_j$. Then $B_1 \subset B_2 \subset \cdots$ are measurable.



$$\text{Theorem 1.32} \quad \Rightarrow \mu\Big(\bigcup_{j=1}^{\infty} B_j\Big) = \lim_{j\to\infty} \mu(B_j).$$

$$\bigcup_{j=1}^{\infty} B_j = \bigcup_{j=1}^{\infty} (A_1 \setminus A_j) = A_1 \setminus \bigcap_{j=1}^{\infty} A_j = A_1 \setminus A$$

$$A_1 = A_j \cup \underbrace{(A_1 \setminus A_j)}_{=B_j} \quad \text{disjoint union} \;\Rightarrow\; \mu(A_1) = \mu(A_j) + \mu(B_j)$$

$$A_1 = A \cup (A_1 \setminus A) \quad \text{disjoint union} \;\Rightarrow\; \mu(A_1) = \mu(A) + \mu(A_1 \setminus A)$$

$$\begin{aligned}\Rightarrow \quad \mu(A) &= \mu(A_1) - \mu(A_1 \setminus A) \quad \text{(here we need } \mu(A_1) < \infty) \\ &= \mu(A_1) - \mu\Big(\bigcup_{j=1}^{\infty} B_j\Big) \\ &= \mu(A_1) - \lim_{j\to\infty} \mu(B_j) \\ &= \mu(A_1) - \lim_{j\to\infty} \big(\mu(A_1) - \mu(A_j)\big) \\ &= \lim_{j\to\infty} \mu(A_j).\end{aligned}$$

□

Remark. The assumption $\mu(A_k) < \infty$ for some $k \in \mathbb{N}$ is necessary. Ex.

$$A_j = \{(x,y) \in \mathbb{R}^2 \colon x > j\}$$
$$A_1 \supset A_2 \supset A_3 \supset \cdots$$
$$m_2(A_j) = \infty \ \forall j$$
$$\bigcap_{j\in\mathbb{N}} A_j = \emptyset \ \Rightarrow \ m_2\Big(\bigcap_{j\in\mathbb{N}} A_j\Big) = 0 \neq \lim_{j\to\infty} m_2(A_j).$$

Remark. (An important application for instance in probability theory) Borel-Cantelli lemma: Let (X, Γ, μ) be a measure space, $A_j \in \Gamma, \ j \in \mathbb{N}$, and

$$A = \{x \in X \colon x \in A_j \text{ for infinitely many } j \in \mathbb{N}\}.$$

Then:

$$\sum_{j=1}^{\infty} \mu(A_j) < \infty \ \Rightarrow \mu(A) = 0.$$

1.34 Non-(Lebesgue-)measurable set in $\mathbb{R}$

Theorem 1.35. *(Vitali, 1905)*

$$\operatorname{Leb}\mathbb{R} \subsetneq \mathcal{P}(\mathbb{R}),$$

in other words, there exists a subset $E \subset \mathbb{R}$ that is not Lebesgue measurable.

An idea is to find a set $B \subset \mathbb{R}, \ 0 < m^*(B) < \infty$, and a decomposition of B

$$B = \bigcup_{i=1}^{\infty} A_i$$

into *disjoint* sets A_i s.t.

$$m^*(A_i) = m^*(A_1) \ \forall i.$$

Then some A_i must be non measurable. A way to guarantee that the sets A_i have the same outer measure is to choose

$$A_i = A + x_i$$

for some (fixed) $A \subset \mathbb{R}$ and $x_i \in \mathbb{R}$, and use the translation invariance of the outer measure m^*.

Proof. Consider the quotient space $\mathbb{R}/\mathbb{Q}$ whose elementys are equivalence classes $E(x),\ x \in \mathbb{R}$.

$$E(x) = E(y) \iff x \sim y \iff x - y \in \mathbb{Q}.$$

We may write $E(x) = x + \mathbb{Q}$. Choose from each equivalence class $E(x),\ x \in \mathbb{R}$, exactly one representative that belongs to the unit interval $[0,1]$. Let A be the set of such chosen points (representatives).

Claim: $A \notin \operatorname{Leb}\mathbb{R}$.

Assume on the contrary: $A \in \operatorname{Leb}\mathbb{R}$.

(i) The sets $A + r,\ r \in \mathbb{Q}$, are disjoint since:

$$\begin{aligned}
x \in (A+r) \cap (A+s),\ r,s \in \mathbb{Q} &\Rightarrow x = a_1 + r \text{ and } x = a_2 + s, \quad a_1, a_2 \in A \\
&\Rightarrow a_1 - a_2 = s - r \in \mathbb{Q} \\
&\Rightarrow a_1 \sim a_2 \Rightarrow E(a_1) = E(a_2) \\
&\Rightarrow a_1 = a_2 \quad \text{(because we choose \underline{exactly} one representative)} \\
&\Rightarrow s = r.
\end{aligned}$$

(ii) $m(A) = 0$ (we use the tranlation invariance: $A \in \operatorname{Leb}\mathbb{R} \Rightarrow A + a \in \operatorname{Leb}\mathbb{R}$ and $m(A) = m(A+a)$):

$$\begin{aligned}
A \subset [0,1] &\Rightarrow A + \frac{1}{n} \subset [0,2] \quad \forall n \in \mathbb{N} \\
&\Rightarrow 2 \geq m\Big(\bigcup_{n=1}^{\infty} (A + \frac{1}{n})\Big) \overset{\text{disjoint}}{=} \sum_{n=1}^{\infty} m(A + \frac{1}{n}) = \sum_{n=1}^{\infty} m(A) \\
&\Rightarrow m(A) = 0.
\end{aligned}$$

(iii) $\mathbb{R} = \bigcup_{r \in \mathbb{Q}} (A + r)$:

$$\begin{aligned}
x \in \mathbb{R} &\Rightarrow \exists a \in E(x) \cap A \Rightarrow x - a = r \in \mathbb{Q},\ a \in A \\
&\Rightarrow x = a + r, \quad a \in A \\
&\Rightarrow x \in A + r.
\end{aligned}$$

(i), (ii) ja (iii) $\Rightarrow$

$$+\infty = m(\mathbb{R}) = \sum_{r \in \mathbb{Q}} m(A + r) = \sum_{r \in \mathbb{Q}} \underbrace{m(A)}_{=0} = 0. \quad \underline{\text{contradiction}}$$

□

Remark. 1. Also in $\mathbb{R}^n$, $\forall n \geq 1$, $\exists$ similar examples, and so

$$\operatorname{Leb}\mathbb{R}^n \subsetneq \mathcal{P}(\mathbb{R}^n).$$

2. If $A \subset \mathbb{R}$ is an arbitrary set s.t. $m^*(A) > 0$, then $\exists\ B \subset A$ s.t. $B \notin \operatorname{Leb}\mathbb{R}$.

2 Measurable mappings

2.1 Measurable mapping

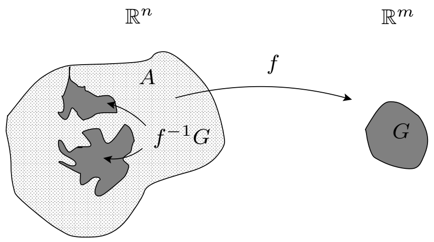
Denote $\dot{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$.

Definition. Let $A \subset \mathbb{R}^n$. A mapping $f\colon A \to \mathbb{R}^m$ is <u>measurable</u> (w.r.t. σ-algebra $\operatorname{Leb}\mathbb{R}^n$) if $f^{-1}G$ is (Lebesgue-)measurable for all open $G \subset \mathbb{R}^m$. A mapping $f\colon A \to \dot{\mathbb{R}}$ is <u>measurable</u> if

(i) $f^{-1}G$ is measurable for all open $G \subset \mathbb{R}^m$,

(ii) $f^{-1}(+\infty)$ is measurable, and

(iii) $f^{-1}(-\infty)$ is measurable.



Remark. 1. $f\colon A \to \mathbb{R}^m$ measurable $\Rightarrow$

$$A = f^{-1}\mathbb{R}^m \subset \mathbb{R}^n \quad \text{is a measurable set.}$$

Similarly $f\colon A \to \dot{\mathbb{R}}$ measurable $\Rightarrow$

$$A = f^{-1}(\mathbb{R}) \cup f^{-1}(+\infty) \cup f^{-1}(+\infty) \subset \mathbb{R}^n \quad \text{is a measurable set.}$$

2. $f\colon A \to \mathbb{R}^m$ measurable, $B \subset A$ measurable $\Rightarrow$ $f|B\colon B \to \mathbb{R}^m$ measurable.
<u>Reason</u>: $G \subset \mathbb{R}^m$ open $\Rightarrow$

$$(f|B)^{-1}(G) = \underbrace{B}_{\text{measurable}} \cap \underbrace{f^{-1}G}_{\text{measurable}}$$

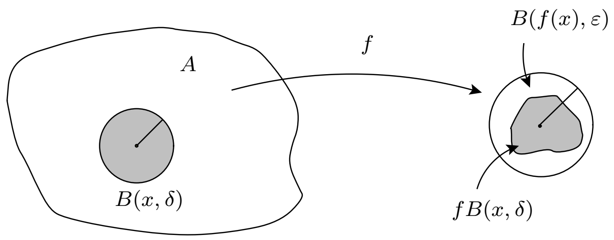
is measurable.

3. Let X be an arbitrary set and $\Gamma \subset \mathcal{P}(X)$ a σ-algebra.
Define: A mapping $f\colon X \to \mathbb{R}$ is <u>measurable</u> (w.r.t. σ-algebra Γ) if $f^{-1}G \in \Gamma$ for all open $G \subset \mathbb{R}$.

Recall A mapping $f\colon A \to \mathbb{R}^m$, $A \subset \mathbb{R}^n$, is <u>continuous</u> at $x \in A$ if $\forall \varepsilon > 0 \;\; \exists \delta = \delta(\varepsilon) > 0$ s.t.

$$f\big(B(x,\delta) \cap A\big) \subset B(f(x), \varepsilon).$$

$f\colon A \to \mathbb{R}^m$ is continous if f is continuous at every $x \in A$.



Fact: $f\colon A \to \mathbb{R}^m$ continuous $\iff$

(2.2) $\qquad f^{-1}G$ is open in A $\forall$ poen $G \subset \mathbb{R}^m$, i.e. $f^{-1}G = A \cap V$, where $V \subset \mathbb{R}^n$ is open.

Theorem 2.3. *A measurable and $f\colon A \to \mathbb{R}^m$ continuous $\Rightarrow$ f measurable.*

Proof.

$$G \subset \mathbb{R}^m \text{ open} \overset{(2.2)}{\Longrightarrow} f^{-1}G \text{ open in } A \Rightarrow \exists \text{ open } V \subset \mathbb{R}^n \text{ s.t.}$$
$$f^{-1}G = \underbrace{A}_{\text{measurable}} \cap \underbrace{V}_{\text{measurable}} \in \operatorname{Leb}\mathbb{R}^n$$
$$\Rightarrow f \text{ measurable.}$$

□

Theorem 2.4. *If $f\colon A \to \mathbb{R}^m$ is measurable, then $f^{-1}B$ is measurable for all Borel sets $B \subset \mathbb{R}^m$.*

Proof. Denote $\Gamma = \{V \subset \mathbb{R}^m\colon f^{-1}V \text{ measurable}\}$. Then Γ is a σ-algebra because:

(1) $f^{-1}\emptyset = \emptyset$ measurable $\Rightarrow$ $\emptyset \in \Gamma$,

(2) $V \in \Gamma \Rightarrow f^{-1}V^c = \underbrace{A}_{\text{measurable}} \setminus \underbrace{f^{-1}V}_{\text{measurable}}$ measurable $\Rightarrow$ $V^c \in \Gamma$,

(3) $V_i \in \Gamma,\ i \in \mathbb{N} \Rightarrow f^{-1}\big(\bigcup_{i\in\mathbb{N}} V_i\big) = \bigcup_{i\in\mathbb{N}} \underbrace{f^{-1}V_i}_{\text{measurable}}$ measurable $\Rightarrow$ $\bigcup_{i\in\mathbb{N}} V_i \in \Gamma$.

Furthermore Γ contains all closed sets because: F closed $\Rightarrow$ F^c open $\Rightarrow$ $f^{-1}F = \big(\underbrace{f^{-1}(F^c)}_{\text{measurable}}\big)^c$ measurable $\Rightarrow$ $F \in \Gamma$.
Hence $\Gamma \supset \operatorname{Bor}\mathbb{R}^m$ (= the smallest σ-algebra that contains all closed sets). □

Corollary 2.5. *If f is measurable, then the preimage $f^{-1}(y)$ of a point y and the preimage $f^{-1}I$ of an interval are measurable.*

Example. Let $E \subset \mathbb{R}^n$ amd $\chi_E\colon \mathbb{R}^n \to \{0,1\}$ the characteristic function of E,

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E,\\ 0, & \text{if } x \notin E. \end{cases}$$

Claim: χ_E measurable function $\iff$ E measurable set.
Proof. $\boxed{\Rightarrow}$ $E = \chi_E^{-1}(1)$ measurable (Cor. 2.5).
$\boxed{\Leftarrow}$ Let E be measurable and $G \subset \mathbb{R}$ ope.

$$\chi_E^{-1}(G) = \begin{cases} \mathbb{R}^n, & \text{if } \{0,1\} \subset G,\\ \emptyset, & \text{if } \{0,1\} \cap G = \emptyset,\\ E, & \text{if } \{0,1\} \cap G = \{1\},\\ E^c, & \text{if } \{0,1\} \cap G = \{0\}. \end{cases}$$

These sets are measurable $\Rightarrow$ χ_E measurable function. □

Theorem 2.6. *Let $f\colon A \to \mathbb{R}^m$ be measurable, $A \subset \mathbb{R}^n$, and $g\colon B \to \mathbb{R}^k$ continuous, where $fA \subset B \subset \mathbb{R}^m$. Then $g \circ f$ is measurable.*

Proof.

$$\left.\begin{array}{l} G \subset \mathbb{R}^k \text{ open} \\ g \text{ continuous} \end{array}\right\} \overset{(2.2)}{\Longrightarrow} g^{-1}G \text{ open in } B$$

$$\Rightarrow \ \exists \ \underline{\text{open}}\ V \subset \mathbb{R}^m \text{ s.t. } g^{-1}G = B \cap V$$

$$\Rightarrow \ (g\circ f)^{-1}G = f^{-1}(g^{-1}G) = f^{-1}(B\cap V) \overset{fA\subset B}{=} f^{-1}(V) \quad \text{measurable.}$$

□

Warning: f and g measurable $\not\Rightarrow$ $g \circ f$ measurable.
If $f\colon A \to \mathbb{R}^m$, then

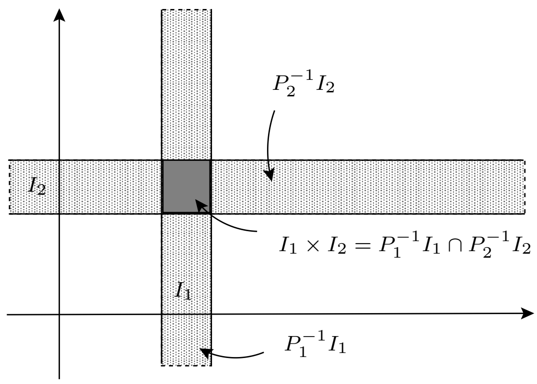
$$f = (f_1, \dots, f_m),\ f(x) = \big(f_1(x), \dots, f_m(x)\big),$$

where

$$f_j\colon A \to \mathbb{R},\ f_j(x) = (P_j \circ f)(x) \text{ and } P_j(y_1, \dots, y_m) = y_j \text{ (= projection onto } j\text{'s coordinate axis).}$$

Theorem 2.7. *$f = (f_1, \dots, f_m)\colon A \to \mathbb{R}^m$ is measurable $\iff$ f_j is measurable $\forall j \in \{1, \dots, m\}$.*

Proof. $\boxed{\Rightarrow}$ If f is measurable, then $f_j = P_j \circ f$ is measurable (Thm. 2.6) since P_j is continuous.
$\boxed{\Leftarrow}$ Suppose that f_j is measurable $\forall j$. Let $G \subset \mathbb{R}^m$ be open.



$$\text{Lindelöf} \ \Rightarrow \ G = \bigcup_{i\in\mathbb{N}} I^{(i)}, \quad I^{(i)} \text{ open } m\text{-interval (cf. proof of Thm. 1.27)}$$

$$I^{(i)} = I_1^{(i)} \times \cdots \times I_m^{(i)} = \bigcap_{j=1}^m P_j^{-1} I_j^{(i)}, \quad I_j^{(i)} \subset \mathbb{R} \text{ open}$$

$$f^{-1}G = \bigcup_{i\in\mathbb{N}} f^{-1}I^{(i)} = \bigcup_{i\in\mathbb{N}} \bigcap_{j=1}^m f^{-1}P_j^{-1}I_j^{(i)} = \bigcup_{i\in\mathbb{N}} \bigcap_{j=1}^m \underbrace{f_j^{-1}I_j^{(i)}}_{\text{measurable}} \quad \text{measurable.}$$

□

Theorem 2.8. *Let $f\colon A \to \dot{\mathbb{R}}$ and $g\colon A \to \dot{\mathbb{R}}$ be measurable. Then their sum and product are measurable (whenever defined). Furthermore, $\lambda f,\ \lambda \in \mathbb{R}$ and $|f|^a,\ a > 0$, are measurable.*

Proof. Sum: Suppose first that $f, g\colon A \to \mathbb{R}$ are measurable. Denote $f + g = u \circ v$, where

$$A \xrightarrow{v} \mathbb{R}^2 \xrightarrow{u} \mathbb{R}, \quad v = (f, g) \quad \text{and} \quad u(x, y) = x + y.$$

$$\left.\begin{array}{r}\text{Thm. 2.7} \;\Rightarrow\; v \text{ measurable} \\ u \text{ continuous}\end{array}\right\} \Rightarrow f + g = u \circ v \text{ measurable.}$$

Note: The case $f, g\colon A \to \mathbb{R}^m$ measurable $\Rightarrow$ $f + g$ measurable follows from Theorem 2.7.

Suppose then that $f, g\colon A \to \dot{\mathbb{R}}$ are measurable. [The sum $f+g$ is defined if there exists no point $x \in A$ such that $\{f(x), g(x)\} = \{+\infty, -\infty\}$.] Denote $f + g = h$. We know that A is measurable (Remark 1.). On the other hand,

$$A = h^{-1}(+\infty) \cup h^{-1}(-\infty) \cup A_0\,, \quad \text{where } A_0 = h^{-1}\mathbb{R}.$$
$$h^{-1}(+\infty) = f^{-1}(+\infty) \cup g^{-1}(+\infty) \;\text{ is measurable.}$$
$$h^{-1}(-\infty) = f^{-1}(-\infty) \cup g^{-1}(-\infty) \;\text{ is measurable.}$$
$$\Rightarrow A_0 \text{ is measurable.}$$

$$f|A_0 \text{ and } g|A_0 \text{ measurable (Remark 2.)} \overset{\text{beginning of proof}}{\Longrightarrow} h^{-1}G \text{ is measurable } \forall G \subset \mathbb{R} \text{ open}$$
$$\Rightarrow h \text{ is measurable.}$$

Product. Similarly (Exerc.)

λf Special case of the product.

$|f|^a$ $|f|^a = u \circ f$, where $u(x) = |x|^a$ continuous if $a > 0$. Thm. 2.6 $\Rightarrow$ $|f|^a$ is measurable. □

From now on we consider only functions $f\colon A \to \dot{\mathbb{R}}$, $A \subset \mathbb{R}^n$.

An important basic criterion:

Theorem 2.9. *Let $A \subset \mathbb{R}^n$ be measurable and $f\colon A \to \dot{\mathbb{R}}$. TFAE (= the following are equivalent)*

(1) f *is measurable;*

(2) $E_a = \{x \in A\colon f(x) < a\}$ *is measurable* $\forall a \in \mathbb{R}$;

(3) $E'_a = \{x \in A\colon f(x) > a\}$ *is measurable* $\forall a \in \mathbb{R}$;

(4) $E''_a = \{x \in A\colon f(x) \le a\}$ *is measurable* $\forall a \in \mathbb{R}$;

(5) $E'''_a = \{x \in A\colon f(x) \ge a\}$ *is measurable* $\forall a \in \mathbb{R}$.

Proof.

$$E'''_a = A \setminus E_a \quad \text{hence (2)} \iff (5)$$
$$E''_a = A \setminus E'_a \quad \text{hence (3)} \iff (4)$$
$$E''_a = \bigcap_{j\in\mathbb{N}} E_{a+1/j} \quad \text{hence (2)} \overset{\text{Thm. 1.18}}{\Longrightarrow} (4)$$
$$E_a = \bigcup_{j\in\mathbb{N}} E''_{a-1/j} \quad \text{hence (4)} \overset{\text{Thm. 1.18}}{\Longrightarrow} (2)$$
$$E_a = f^{-1}\big(\underbrace{(-\infty, a)}_{\text{open}}\big) \cup f^{-1}(-\infty) \quad \text{hence (1)} \Rightarrow (2)$$

Suppose that (2) holds [and thus also (3),(4),(5)] Claim: (1) holds, that is, f is measurable.
Proof: Let $G \subset \mathbb{R}$ be open.

$$G = \bigcup_{j\in\mathbb{N}} I_j, \quad I_j = (a_j, b_j) \text{ open interval (Lindelöf)}$$
$$f^{-1}G = \bigcup_{j\in\mathbb{N}} f^{-1}I_j, \quad f^{-1}I_j = \{x\colon a_j < f(x) < b_j\} = E'_{a_j} \cap E_{b_j} \text{ measurable}$$
$$\Rightarrow f^{-1}G \text{ measurable}$$
$$f^{-1}(+\infty) = \bigcap_{j\in\mathbb{N}} E'_j \text{ measurable}$$
$$f^{-1}(-\infty) = \bigcap_{j\in\mathbb{N}} E_{-j} \text{ measurable}$$
$$\Rightarrow f \text{ measurable.}$$

□

Remark. The assumption "A measurable" is necessary in Theorem 2.9. Example: Let A be non-measurable (Thm. 1.35) and $x_0 \in A$. Define $f\colon A \to \dot{\mathbb{R}}$,

$$f(x) = \begin{cases} +\infty & \text{if } x \in A \setminus \{x_0\}, \\ -\infty & \text{if } x = x_0. \end{cases}$$

Then $E_a = \{x \in A\colon f(x) < a\} = \{x_0\}$ is measurable $\forall a \in \mathbb{R}$, thus (2) holds but f can not be measurable (since A non-measurable), that is (1) does not hold.

Example. Claim: $f\colon \mathbb{R} \to \mathbb{R}$ measurable $\iff$

$$\begin{cases} (1) & f^2 \text{ measurable function,} \\ (2) & E = \{x\colon f(x) > 0\} \text{ measurable set.} \end{cases}$$

Proof: $\boxed{\Leftarrow}$ Denote $E_a = \{x\colon f(x) < a\}$. We must prove E_a is measurable $\forall a \in \mathbb{R}$ (Theorem 2.9).

(i) Let $a > 0$.

$$f(x) < a \iff f(x)^2 < a^2 \text{ or } f(x) \le 0, \text{ hence}$$
$$E_a = \underbrace{\{x\colon f^2(x) < a^2\}}_{\text{measurable (1)}} \cup \underbrace{E^c}_{\text{measurable (2)}} \quad \text{measurable.}$$

(ii) Let $a \le 0$.

$$f(x) < a \iff f(x)^2 > a^2 \text{ and } f(x) \le 0, \text{ hence}$$
$$E_a = \underbrace{\{x\colon f^2(x) > a^2\}}_{\text{measurable (1)}} \cap \underbrace{E^c}_{\text{measurable (2)}} \quad \text{measurable.}$$

Theorem 2.9 $\Rightarrow$ f is measurable.

$\boxed{\Rightarrow}$ f measurable $\overset{\text{Thm. 2.8}}{\Rightarrow}$ $f^2 = f \cdot f$ is measurable. Similarly: f measurable $\overset{\text{Thm. 2.9}}{\Rightarrow}$ E measurable. □

Remark. f^2 measurable $\not\Rightarrow$ f measurable. <u>Reason</u>: Let $E \subset \mathbb{R}$ be non-measurable and $f\colon \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} 1, & \text{if } x \in E, \\ -1, & \text{if } x \in E^c. \end{cases}$$

Then f^2 is measurable as a constant function $f^2(x) \equiv 1$ but $\{x\colon f(x) > 0\} = E$ is non-measurable set. $\overset{\text{Thm. 2.9}}{\Longrightarrow}$ f non-measurable.

2.10 $\limsup$ and $\liminf$ of a sequence

Definition. Let $a_1, a_2, \ldots$ be a sequence in $\dot{\mathbb{R}}$. Denote

$$b_k = \sup_{i \geq k} a_i, \quad c_k = \inf_{i \geq k} a_i. \quad (b_k, c_k \in \dot{\mathbb{R}} \text{ allowed})$$

Then

$$\begin{aligned} &b_1 \geq b_2 \geq \cdots \geq b_k \geq b_{k+1} \geq \cdots \quad \text{and} \\ &c_1 \leq c_2 \leq \cdots \leq c_k \leq c_{k+1} \leq \cdots \quad (\sup / \inf \text{ taken over a smaller set}) \end{aligned}$$

$\Rightarrow$ $\exists$ limits

$$\lim_{k \to \infty} b_k = \inf_{k \in \mathbb{N}} b_k = \beta \quad \text{and} \quad \lim_{k \to \infty} c_k = \sup_{k \in \mathbb{N}} c_k = \gamma \quad (\pm\infty \text{ allowed}).$$

Denote

$$\begin{aligned} \beta &= \limsup_{i \to \infty} a_i \quad \text{or} \quad \overline{\lim_{i \to \infty}}\, a_i \quad \text{"upper limit" or "limes superior"} \\ \gamma &= \liminf_{i \to \infty} a_i \quad \text{or} \quad \underline{\lim_{i \to \infty}}\, a_i \quad \text{"lower limit" or "limes inferior".} \end{aligned}$$

Thus

$$\begin{aligned} \limsup_{i \to \infty} a_i &= \lim_{k \to \infty} \big(\sup_{i \geq k} a_i\big) = \inf_{k \in \mathbb{N}} \big(\sup_{i \geq k} a_i\big), \\ \liminf_{i \to \infty} a_i &= \lim_{k \to \infty} \big(\inf_{i \geq k} a_i\big) = \sup_{k \in \mathbb{N}} \big(\inf_{i \geq k} a_i\big). \end{aligned}$$

Remark. (a_i) a sequence in $\dot{\mathbb{R}}$ $\Rightarrow$ $\limsup_{i \to \infty} a_i$ and $\liminf_{i \to \infty} a_i$ <u>always</u> exist ($\in \dot{\mathbb{R}}$) and are unique.

Example. (1) $\infty, -\infty, \infty, -\infty, \ldots$; $b_k = \infty\ \forall k$, $c_k = -\infty\ \forall k$ $\Rightarrow$ $\beta = \infty$, $\gamma = -\infty$

(2) $1, 2, 3, 4, \ldots$; $b_k = \infty\ \forall k$, $c_k = k\ \forall k$ $\Rightarrow$ $\beta = \infty = \gamma$

(3) $0, 1, 0, 1, 0, 1, \ldots$; $b_k = 1\ \forall k$, $c_k = 0\ \forall k$ $\Rightarrow$ $\beta = 1$, $\gamma = 0$

(4) $0, -1, 0, -2, 0, -3, \ldots$; $b_k = 0\ \forall k$, $c_k = -\infty\ \forall k$ $\Rightarrow$ $\beta = 0$, $\gamma = -\infty$.

Theorem 2.11. *(i)* $\liminf_{i \to \infty} a_i \leq \limsup_{i \to \infty} a_i$,

(ii) $a_i \leq M\ \forall i \geq i_0$ $\Rightarrow$ $\limsup_{i \to \infty} a_i \leq M$,

(iii) $a_i \geq m\ \forall i \geq i_0$ $\Rightarrow$ $\liminf_{i \to \infty} a_i \geq m$.

Proof. (i) $c_k \le b_k \;\Rightarrow\; \gamma = \lim_{k\to\infty} c_k \le \lim_{k\to\infty} b_k = \beta$,

(ii) $b_k \le M \ \forall k \ge i_0 \;\Rightarrow\; \beta = \lim_{k\to\infty} b_k \le M$,

(iii) $c_k \ge m \ \forall k \ge i_0 \;\Rightarrow\; \gamma = \lim_{k\to\infty} c_k \ge m$.

□

Theorem 2.12. *Let (a_i) be a sequence in $\dot{\mathbb{R}}$. Then*

$$\exists \lim_{i\to\infty} a_i \ (\in \dot{\mathbb{R}}) \iff \liminf_{i\to\infty} a_i = \limsup_{i\to\infty} a_i \quad (\in \dot{\mathbb{R}}).$$

In this case

$$\lim_{i\to\infty} a_i = \liminf_{i\to\infty} a_i = \limsup_{i\to\infty} a_i \quad (\pm\infty \quad \textit{allowed}).$$

Proof. $\boxed{\Rightarrow}$ Suppose that $\exists \alpha = \lim_{i\to\infty} a_i$.

(a1) $\alpha \in \mathbb{R}$

$$\begin{gathered}
\varepsilon > 0 \;\Rightarrow\; \exists i_0 \text{ s.t. } \alpha - \varepsilon < a_i < \alpha + \varepsilon \ \forall i \ge i_0 \\
\Rightarrow\; \alpha - \varepsilon \le c_{i_0} \le \gamma \le \beta \le b_{i_0} \le \alpha + \varepsilon \\
\varepsilon \quad \text{arbotrary} \quad \Rightarrow\; \gamma = \beta
\end{gathered}$$

(a2) $\alpha = \infty$

$$\begin{gathered}
M \in \mathbb{R} \;\Rightarrow\; \exists i_0 \text{ s.t. } a_i > M \ \forall i \ge i_0 \\
\Rightarrow\; M \le c_{i_0} \le \gamma \le \beta \\
M \quad \text{arbitrary} \quad \Rightarrow\; \gamma = \beta = \infty
\end{gathered}$$

(a3) $\alpha = -\infty$ similarly.

$\boxed{\Leftarrow}$ Suppose that $\beta = \gamma \overset{\text{denote}}{=} \alpha$.

(b1) $\alpha \in \mathbb{R}$

$$\begin{gathered}
\varepsilon > 0 \;\Rightarrow\; \exists k_1 \text{ s.t. } b_k < \alpha + \varepsilon \ \forall k \ge k_1 \\
\exists k_2 \text{ s.t. } c_k > \alpha - \varepsilon \ \forall k \ge k_2 \\
k \ge \max\{k_1, k_2\} \Rightarrow\; \alpha - \varepsilon < c_k \le a_k \le b_k < \alpha + \varepsilon \\
\varepsilon \quad \text{arbitrary} \quad \Rightarrow\; \alpha = \lim_{k\to\infty} a_k
\end{gathered}$$

(b2) $\alpha = \infty$

$$\begin{gathered}
M \in \mathbb{R} \;\Rightarrow\; \exists k_0 \text{ s.t. } c_k > M \ \forall k \ge k_0 \\
\Rightarrow\; a_k \ge c_k > M \ \forall k \ge k_0 \\
\Rightarrow\; \lim_{k\to\infty} a_k = \infty
\end{gathered}$$

(b3) $\alpha = -\infty$ similarly. □

2.13 Measurablity of limit function

Theorem 2.14. *Let $f_j\colon A \to \dot{\mathbb{R}}$, $j \in \mathbb{N}$, be measurable. Then the functions*

$$\sup_{j\in\mathbb{N}} f_j, \quad \inf_{j\in\mathbb{N}} f_j, \quad \limsup_{j\to\infty} f_j, \quad \liminf_{j\to\infty} f_j$$

are measurable. If $\exists f = \lim_{j\to\infty} f_j$, then f is measurable.

Remark. These functions are defined pointwise $\forall x \in A$. For instance, the value of the function $\sup_{j\in\mathbb{N}} f_j$ at a point $x \in A$ is $\sup_{j\in\mathbb{N}} f_j(x) \in \dot{\mathbb{R}}$.

Proof. Denote $g(x) = \sup_{j\in\mathbb{N}} f_j(x),\ x \in A$. For all $a \in \mathbb{R}$:

(2.15)

$$\{x \in A\colon g(x) \le a\} \overset{(*)}{=} \bigcap_{j\in\mathbb{N}} \overbrace{\{x \in A\colon f_j(x) \le a\}}^{\text{measurable}} \quad \text{is measurable} \ \Rightarrow\ g = \sup_{j\in\mathbb{N}} f_j \text{ is measurable.}$$

$$\big(\ (*):\ g(x) \le a \iff f_j(x) \le a\ \forall j \in \mathbb{N}\big)$$

(2.16)

$$\inf_{j\in\mathbb{N}} f_j = -\sup_{j\in\mathbb{N}}(-f_j) \quad \text{is measurable,}$$

$$\limsup_{j\to\infty} f_j = \inf_{k\in\mathbb{N}}\big(\sup_{j\ge k} f_j\big) \quad \text{is measurabel [(2.15), (2.16)],}$$

$$\liminf_{j\to\infty} f_j = \sup_{k\in\mathbb{N}}\big(\inf_{j\ge k} f_j\big) \quad \text{is measurable [(2.15), (2.16)].}$$

$$\text{If } \exists f = \lim_{j\to\infty} f_j, \text{ then } \lim_{j\to\infty} f_j \overset{\text{Thm. 2.12}}{=} \limsup_{j\to\infty} f_j \quad \text{is measurable.}$$

□

Almost every(where) (abbreviated a.e.) = except a set of measure zero.
Example:

(a) a.e. real number is irrational, because $m(\mathbb{Q}) = 0$.

(b) $e^{-jx^2} \xrightarrow{j\to\infty} 0$ for a.e. $x \in \mathbb{R}$ since $m(\{0\}) = 0$.

Theorem 2.17. *Let $f, g\colon A \to \dot{\mathbb{R}}$. Suppose that f is measurable and $g = f$ a.e. Then g is measurable.*

Proof. $f, g\colon A \to \dot{\mathbb{R}}$ and $f(x) = g(x)\ \forall x \in A \setminus A_0$, where $A_0 \subset A,\ m(A_0) = 0$. Let $a \in \mathbb{R}$. Denote

$$E_a = \underbrace{\{x \in A\colon f(x) < a\}}_{\text{measurable}} \quad \text{and} \quad F_a = \{x \in A\colon g(x) < a\}.$$

$$F_a = \big(F_a \cap A_0\big) \cup \big(F_a \setminus A_0\big),$$

$$m^*\big(F_a \cap A_0\big) \le m^*(A_0) = 0 \ \Rightarrow\ F_a \cap A_0 \quad \text{is measurable.}$$

$$F_a \setminus A_0 = E_a \setminus A_0 \quad \text{is measurable}$$

$$\Rightarrow\ F_a \quad \text{Is measurable.}$$

□

Remark. Hence sets of measure zero do not affect on measurability $\Rightarrow$ we may talk about measurability of functions that are defined only a.e.

Theorem 2.18. *Let $f_j\colon A \to \dot{\mathbb{R}}$, $j \in \mathbb{N}$, be measurable and $f_j \to f$ a.e. Then f is measurable.*

Proof. $f = \limsup_{j\to\infty} f_j$ a.e. □

Example. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ and $\exists f'(x)\ \forall x \in \mathbb{R}$.
Claim: f' is measurable.
Proof: Denote

$$g_n(x) = \frac{f(x+1/n) - f(x)}{1/n}, \quad \text{hence} \quad f'(x) = \lim_{n\to\infty} g_n(x).$$

$\exists\ f'(x)\ \forall x \in \mathbb{R} \ \Rightarrow\ f$ continuous and therefore measurable $\Rightarrow\ g_n$ measurable (Thm. 2.8)

$$\overset{\text{Thm. 2.14}}{\Longrightarrow}\ f' \text{ measurable.}$$

□

3 Lebesgue integral

3.1 Simple functions

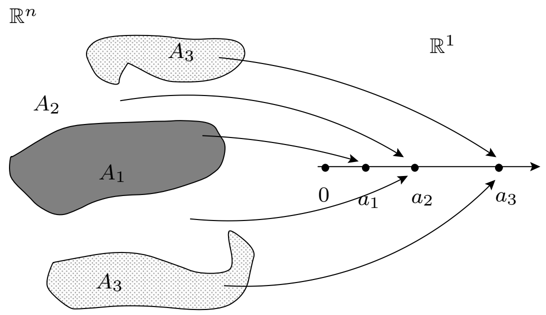
Definition. A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is *simple* if

(1) f is measurable,

(2) $f \geq 0 \quad (f(x) \geq 0\ \forall x \in \mathbb{R}^n)$,

(3) f takes only finitely many values.

Denote $Y = \{f \mid f\colon \mathbb{R}^n \to \mathbb{R} \text{ simple}\}$ (or Y_n).



Remark. 1. $f \in Y \ \Rightarrow\ f(x) \neq \infty\ \forall x$.

2. $f \in Y,\ E \in \operatorname{Leb}\mathbb{R}^n \ \Rightarrow\ f\chi_E \in Y$.

Let $f \in Y$ and let $a_1, \ldots, a_k \in [0, +\infty)$ be the values of f. Then

$$A_i = f^{-1}(a_i) \quad \text{are measurable and disjoint,}\ \mathbb{R}^n = \bigcup_{i=1}^{k} A_i$$

and

$$\boxed{f = \sum_{i=1}^{k} a_i \cdot \chi_{A_i}} \quad \text{is the standard representation of } f.$$

Definition. Let $f \in Y$ and $f = \sum_{i=1}^{k} a_i \cdot \chi_{A_i}$ its standard representation. Then the *integral* of f (over $\mathbb{R}^n$) is

$$I(f) = \sum_{i=1}^{k} a_i m(A_i). \qquad \text{(recall } 0 \cdot \infty = 0\text{)}$$

If $E \subset \mathbb{R}^n$ is measurable, then the integral of f over E is

$$I(f, E) = I(f\chi_E).$$

In particular:

$I(f) = I(f, \mathbb{R}^n)$,

$0 \leq I(f, E) \leq \infty$,

$E \in \operatorname{Leb} \mathbb{R}^n \;\Rightarrow\; I(\chi_E) = m(E)$.

Theorem 3.2. *If $f \in Y$ and $\sum_{i=1}^{k} a_i \cdot \chi_{A_i}$ is the standard representation of f, then*

$$I(f, E) = \sum_{i=1}^{k} a_i m(A_i \cap E).$$

Proof. Omitted. □

Theorem 3.3. *Let $E_j,\ j \in \mathbb{N}$, be measurable and disjoint sets and let $E = \bigcup_{j \in \mathbb{N}} E_j$. If $f \in Y$, then*

$$I(f, E) = \sum_{j \in \mathbb{N}} I(f, E_j).$$

Proof. Let $f = \sum_{i=1}^{k} a_i \chi_{A_i}$ be the standard representation.

$$\text{L. } 3.2 \;\Rightarrow\; I(f, E) = \sum_{i=1}^{k} a_i m(A_i \cap E).$$

Since $A_i \cap E = \bigcup_{j \in \mathbb{N}} (A_i \cap E_j)$, then (by the countable additivity Thm. 1.18)

$$m(A_i \cap E) = \sum_{j \in \mathbb{N}} m(A_i \cap E_j) \quad \forall i = 1, \ldots, k$$

$$\Rightarrow I(f; E) = \sum_{i=1}^{k} a_i \sum_{j \in \mathbb{N}} m(A_i \cap E_j) = \sum_{j \in \mathbb{N}} \sum_{i=1}^{k} a_i m(A_i \cap E_j)$$

$$\overset{3.2}{=} \sum_{j \in \mathbb{N}} I(f, E_j).$$

□

Remark. Clearly $I(f, \emptyset) = I(f\chi_\emptyset) = I(0) = 0$, and therefore by Thm. 3.3 the mapping

$$\operatorname{Leb} \mathbb{R}^n \to [0, +\infty], \quad E \mapsto I(f, E)$$

is a measure for every (fixed) $f \in Y$.

Convergence theorem 1.32 $\Rightarrow$

Corollary 3.4. *If $f \in Y$ and $E_1 \subset E_2 \subset \cdots$ are measurable, then*

$$I\big(f, \cup_{j=1}^{\infty} E_j\big) = \lim_{j\to\infty} I(f, E_j).$$

Theorem 3.5. *Let $f, g \in Y$, E measurable, and $a \geq 0$ a constant. Then*

(i) $f+g \in Y$ and $I(f+g, E) = I(f,E) + I(g,E)$;

(ii) $af \in Y$ and $I(af, E) = aI(f,E)$.

Proof. (i): Clearly $f+g \in Y$.

(a) Let $E = \mathbb{R}^n$ and

$$f = \sum_{j=1}^{k} a_j \chi_{A_j}, \quad g = \sum_{i=1}^{\ell} b_i \chi_{B_i}$$

the standard representation. Then

$$(f+g)\chi_{A_i \cap B_j} = (a_i + b_j)\chi_{A_i\cap B_j} \quad \forall i,j \quad \overset{3.2}{\Longrightarrow}$$

$$\text{(3.6)} \qquad \begin{cases} I(f+g, A_i \cap B_j) &= (a_i + bj)m(A_i \cap B_j) = a_i m(A_i \cap B_j) + b_j m(A_i \cap B_j) \\ &= I(f, A_i \cap B_j) + I(g, A_i \cap B_j) \end{cases}$$

$\mathbb{R}^n$ = disjoint union of sets $A_i \cap B_j$. Theorem 3.3 $\Rightarrow$

$$\begin{aligned} I(f+g) &\overset{3.3}{=} \sum_{i,j} I(f+g, A_i\cap B_j) \overset{(3.6)}{=} \sum_{i,j} I(f, A_i \cap B_j) + \sum_{i,j} I(g, A_i \cap B_j) \\ &\overset{3.3}{=} I(f) + I(g) \end{aligned}$$

(b) E arbitrary.

$$\begin{aligned} I(f+g,E) &= I\big((f+g)\chi_E\big) = I(f\chi_E + g\chi_E) = I(f\chi_E) + I(g\chi_E) \\ &= I(f,E) + I(g,E). \end{aligned}$$

(ii): $af \in Y$ clear.

$$a = 0 \;\Rightarrow\; I(af, E) = 0 = aI(f,E).$$

Let $a > 0$ and $f = \sum_{i=1}^{k} a_i \chi_{A_i}$ the standard representation.

$$af = \sum_{i=1}^{k} a a_i \chi_{A_i} \quad \text{standard representation.}$$

$$I(af, E) = \sum_{i=1}^{k} a a_i m(A_i \cap E) = a \sum_{i=1}^{k} a_i m(A_i \cap E) = aI(f,E).$$

$\square$

Monotonicity properties.

Theorem 3.7. *(1) E measurable and $f, g \in Y$, $f \leq g$ (i.e. $f(x) \leq g(x)\ \forall x$) $\Rightarrow$ $I(f,E) \leq I(g,E)$;*

(2) $E \subset F$ measurable, $f \in Y$ $\Rightarrow$ $I(f,E) \leq I(f,F)$;

(3) $f \in Y$, $m(E) = 0$ $\Rightarrow$ $I(f,E) = 0$.

Proof. (1): $g = f + (g - f)$, where $g - f \geq 0$ and $g - f \in Y$. Theorem 3.5 $\Rightarrow$

$$I(g,E) \overset{3.5}{=} I(f,E) + \underbrace{I(g-f,E)}_{\geq 0} \geq I(f,E).$$

(2):

$$\left.\begin{aligned} E \subset F \ \Rightarrow\ 0 \leq \chi_E \leq \chi_F \\ f \in Y \end{aligned}\right\} \Rightarrow\ f\chi_E \leq f\chi_F \quad (\in Y)$$

$$\Rightarrow\ I(f,E) = I(f\chi_E) \overset{(1)}{\leq} I(f\chi_F) = I(f,F).$$

(3): If $f = \sum_{i=1}^k a_i \chi_{A_i}$ is the standard representation, then

$$I(f,E) = \sum_{i=1}^k a_i \underbrace{m(A_i \cap E)}_{=0} = 0 \quad \text{since } A_i \cap E \subset E \text{ and } m(E) = 0.$$

□

3.8 Lebesgue integral, $f \geq 0$

Theorem 3.9. *Let $f\colon \mathbb{R}^n \to \dot{\mathbb{R}}$ be measurable and $f \geq 0$. Then $\exists$ an increasing sequence of simple functions $f_j \in Y$, $f_1 \leq f_2 \leq \cdots$, s.t. $f(x) = \lim_{j\to\infty} f_j(x)\ \forall x \in \mathbb{R}^n$.*

Proof. Define $f_j\colon \mathbb{R}^n \to \dot{\mathbb{R}}$ as follows: Divide $[0,j)$ into disjoint half open intervals $I_1, \ldots, I_k$, whose length is $1/2^j$, i.e.

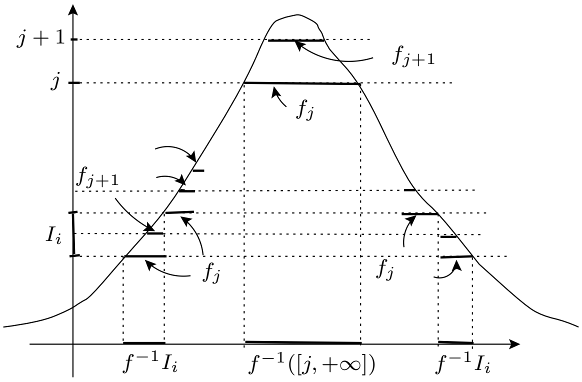
$$I_i = [(i-1)2^{-j}, i2^{-j}), \quad i = 1, \ldots, k = j2^j.$$

Define

$$f_j(x) = \begin{cases} (i-1)2^{-j}, & \text{if } x \in f^{-1}I_i, \quad (\text{i.e. } (i-1)2^{-j} \leq f(x) < i2^{-j}) \\ j, & \text{if } x \in f^{-1}[j,+\infty] \quad (\text{i.e. } f(x) \geq j). \end{cases}$$

$$\left.\begin{aligned} f \text{ measurable } \Rightarrow\ f^{-1}(I_i) \text{ measurable and} \\ f^{-1}[j,+\infty] \text{ measurable.} \\ f_j \geq 0, \text{ takes only finitely many values} \end{aligned}\right\} \Rightarrow\ f_j \in Y,\ j = 1, 2, \ldots$$

Construction $\Rightarrow$ $f_j \leq f_{j+1}$ (see the picture).



Claim: $f_j(x) \to f(x)\ \forall x \in \mathbb{R}^n$.

(a): $f(x) < +\infty \ \Rightarrow\ \exists j_0 > f(x)$. If $j \geq j_0$, then

$$(i-1)2^{-j} \leq f(x) < i2^{-j} \text{ for some } i \in \{1,\ldots,j2^j\}$$
$$\Rightarrow\ f_j(x) = (i-1)2^{-j} \leq f(x) < i2^{-j} = f_j(x) + 2^{-j} \ \Rightarrow\ f(x) - 2^{-j} < f_j(x) \leq f(x)$$
$$\Rightarrow\ \lim_{j\to\infty} f_j(x) = f(x).$$

(b): $f(x) = +\infty \ \Rightarrow\ f_j(x) = j\ \forall j \ \Rightarrow\ f_j(x) \to +\infty = f(x)$. □

Definition. Let $f\colon \mathbb{R}^n \to \dot{\mathbb{R}}$ be measurable and $f \geq 0$. Then the *(Lebesgue) integral* of f over $\mathbb{R}^n$ is

$$\int f = \sup\{I(\varphi)\colon \varphi \in Y,\ \varphi \leq f\}.$$

If $E \subset \mathbb{R}^n$ is measurable, then the integral of f over E is

$$\int_E f = \int f\chi_E. \tag{3.10}$$

Denote also

$$\int_E f = \int_E f\,dm = \int_E f(x)\,dm(x), \quad m = n\text{-dimensional Lebesgue measure.}$$

If $n = 1$ and $E = [a,b]$, we denote $\int_E f = \int_a^b f = \int_a^b f(x)\,dx$.

Convention. If $f\colon A \to \dot{\mathbb{R}}$ and $E \subset A$, then we define $f\chi_E\colon \mathbb{R}^n \to \dot{\mathbb{R}}$,

$$f\chi_E(x) = \begin{cases} f(x), & \text{if } x \in E,\\ 0, & \text{if } x \notin E.\end{cases}$$

Then (3.10) defines $\int_E f$ for all measurable $f\colon A \to \dot{\mathbb{R}}$ and measurable $E \subset A$.

Theorem 3.11. *$f \in Y$ and E measurable $\Rightarrow$ $I(f,E) = \int_E f$.*

Proof. We may assume $E = \mathbb{R}^n$ (otherwise replace f by $f\chi_E \in Y$).

(a) $f \leq f \ \Rightarrow\ I(f) \leq \int f$.

(b) $\varphi \in Y,\ \varphi \leq f \ \overset{\text{L. 3.7 (1)}}{\Longrightarrow}\ I(\varphi) \leq I(f) \ \Rightarrow\ \int f \leq I(f)$.

□

Basic properties of integrals.

Theorem 3.12. *Suppose that the functions below are non-negative and measurable and the sets are measurable subsets of* $\mathbb{R}^n$.

(1) $f \le g \;\Rightarrow\; \int_E f \le \int_E g$

(2) $A \subset B \;\Rightarrow\; \int_A f \le \int_B g$

(3) $f(x) = 0\ \forall x \in E \;\Rightarrow\; \int_E f = 0$

(4) $m(E) = 0 \;\Rightarrow\; \int_E f = 0$

(5) $0 \le a < \infty \;\Rightarrow\; \int_E af = a\int_E f.$

Proof. (1): Let $E = \mathbb{R}^n$, $\varphi \in Y$, $\varphi \le f \;\Rightarrow\; \varphi \le g \;\Rightarrow$

$$I(\varphi) \le \int g \overset{\text{sup}}{\Longrightarrow} \int f \le \int g.$$

$E \in \operatorname{Leb}\mathbb{R}^n \;\Rightarrow\; f\chi_E \le g\chi_E$ in $\mathbb{R}^n \overset{(1)}{\Longrightarrow}$

$$\int_E f = \int f\chi_E \le \int g\chi_E = \int_E g.$$

(2): $f\chi_A \le f\chi_B$ ja (1) $\Rightarrow$ claim.
(3): $f\chi_E = 0 \;\Rightarrow\; \int_E f = I(0) = 0$.
(4): Let $\varphi \in Y$, $\varphi \le f\chi_E$. Since $\varphi|\mathbb{R}^n \setminus E = 0$, then $\varphi = \varphi\chi_E$ and

$$I(\varphi) = I(\varphi, E) \overset{3.7\,(3)}{=} 0 \overset{\text{sup}}{\Longrightarrow} \int_E f = 0.$$

(5): If $a = 0$, both sides are zero. Let $a > 0$, $\varphi \in Y$, $\varphi \le f\chi_E \;\Rightarrow\; a\varphi \le af\chi_E \;\Rightarrow$

$$\int_E af \ge I(a\varphi) \overset{3.5\,\text{(ii)}}{=} aI(\varphi) \;\Rightarrow\; \int_E af \ge a\int_E f.$$
$$f = \frac{1}{a}(af) \;\Rightarrow\; \int_E f = \int_E \frac{1}{a}(af) \overset{\text{yllä}}{\ge} \frac{1}{a}\int_E af \;\Rightarrow\; a\int_E f \ge \int_E af.$$

□

Relation to the Riemann integral.

Theorem 3.13. *Let* $E \subset \mathbb{R}^n$ *be bounded and* $f\colon E \to \mathbb{R}$ *measurable,* $f \ge 0$. *If* f *is Riemann integrable over* E*, then the*

$$\textit{(Riemann integral)}\quad (\mathrm{R})\int_E f = \int_E f \quad \textit{(Lebesgue integral)}.$$

This is the case, for example, when E *is a closed* n*-interval and* f *continuous.*

Proof. Choose a closed n-interval $I \supset E$. By definition

$$\text{(R)}\int_E f = \text{(R)}\int_I f\chi_E \quad \text{and} \quad \int_E f = \int f\chi_E = \int_I f\chi_E,$$

we may assume that $E = I$ (by replacing f with $f\chi_E$). Let $D = \{I_1, \dots, I_k\}$ be a partition of I into half-open disjoint intervals. Denote

$$\begin{aligned} g_i &= \inf_{x\in I_i} f(x),\ \bar{g}_i = \inf_{x\in \bar{I}_i} f(x) \quad \Rightarrow\ \bar{g}_i \le g_i \quad \text{and} \\ G_i &= \sup_{x\in I_i} f(x),\ \bar{G}_i = \sup_{x\in \bar{I}_i} f(x) \quad \Rightarrow\ \bar{G}_i \ge G_i. \end{aligned}$$

The (Riemann) lower sum is

$$m_D = \sum_{i=1}^{k} \bar{g}_i \ell(I_i) \le \sum_{i=1}^{k} g_i m(I_i) = I(\varphi),$$

where $\varphi = \sum_{i=1}^{k} g_i \chi_{I_i} \in Y$. Similarly the upper sum is

$$M_D = \sum_{i=1}^{k} \bar{G}_i \ell(I_i) \ge \sum_{i=1}^{k} G_i m(I_i) = I(\psi),$$

where $\psi = \sum_{i=1}^{k} G_i \chi_{I_i} \in Y$. Clearly $\varphi \le f \le \psi$, and therefore

$$\text{(3.14)} \qquad m_D \le\ I(\varphi) \overset{\sup}{\le} \int_E f \overset{f\le\psi}{\le} \int_E \psi = I(\psi) \le M_D.$$

Suppose that f is Riemann integrable over E. Then $\forall \varepsilon > 0\ \exists$ a partition D as above s.t.

$$\text{(3.15)} \qquad m_D \le \text{(R)}\int_E f \le M_D \text{ (always)} \quad \text{and} \quad 0 \le M_D - m_D < \varepsilon.$$

Letting $\varepsilon \to 0$ we obtain from (3.14) and (3.15) $\Rightarrow$

$$\text{(R)}\int_E f = \int_E f.$$

□

Remark. The case where E is unbounded (improper Riemann integral) is more complicated. A counterpart of Theorem 3.13 holds if $f \ge 0$, but not in general.

The Lebesgue integral is more general than the Riemann integral:

Example. Let $f = \chi_{\mathbb{Q}}$, $\mathbb{Q}$ = rational numbers. Then f is simple because $f^{-1}(1) = \mathbb{Q}$ and $f^{-1}(0) = \mathbb{R} \setminus \mathbb{Q}$ are measurable.

$$\int_E f = m(E \cap \mathbb{Q}) = 0 \quad \forall \text{ measurable } E \subset \mathbb{R}.$$

On the other hand, f is not Riemann integrable over any interval $[a, b]$, $a < b$,: Let $D = \{I_1, \dots, I_k\}$ be a partition of $[a, b]$ into subintervals. Every I_i contains both rational and irrational numbers. Hence

$$\Rightarrow\ m_D = \sum_i 0 \cdot \ell(I_i) = 0 \text{ and } M_D = \sum_i 1 \cdot \ell(I_i) = b - a.$$

Theorem 3.16. *Let $f\colon E \to \dot{\mathbb{R}}$ be measurable, $f \ge 0$ and $\int_E f < \infty$. Then $f(x) < \infty$ for a.e. $x \in E$.*

Proof. Denote $A = \{x \in E\colon f(x) = \infty\}$ (measurable set since f is measurable).

$$\begin{gathered} f(x) \ge j \quad \forall x \in A,\ j = 1,2,\ldots \ \Rightarrow\ j\chi_A \le f\chi_E \quad \forall j \\ \Rightarrow\ \int_E f \ge I(j\chi_A) = jm(A) \quad \forall j \\ 0 \le m(A) \le \frac{1}{j} \underbrace{\int_E f}_{<\infty} \xrightarrow{j\to\infty} 0 \ \Rightarrow\ m(A) = 0. \end{gathered}$$

□

Monotone convergence theorem.

Theorem 3.17. (MCT) *Let $f_j\colon E \to \dot{\mathbb{R}}$ be measurable and*

$$0 \le f_1 \le f_2 \le \cdots \le f_j \le f_{j+1} \le \cdots .$$

Then

$$\lim_{j\to\infty} \int_E f_j = \int_E \lim_{j\to\infty} f_j \quad (+\infty \textit{ aloowed}).$$

Proof. $f_j \le f_{j+1} \ \Rightarrow\ \int_E f_j \le \int_E f_{j+1} \ \Rightarrow\ \exists$ a limit $\lim_{j\to\infty} \int_E f_j = a$ ($\in [0,\infty]$). Similarly, $\exists\ f = \lim_{j\to\infty} f_j$ that is measurable (Thm. 2.14).

$$f_j \le f \ \Rightarrow\ \int_E f_j \le \int_E f \ \Rightarrow\ a \le \int_E f.$$

Need to prove: $\int_E f \le a$.

May assume: $E = \mathbb{R}^n$ (otherwise replace $f_j,\ f$ by functions $f_j\chi_E,\ f\chi_E$ (note: $f_j\chi_E \nearrow f\chi_E$)). Let $0 < b < 1,\ \varphi \in Y,\ \varphi \le f$. Denote

$$E_j = \{x \in \mathbb{R}^n\colon f_j(x) \ge b\varphi(x)\} = \{x \in \mathbb{R}^n\colon \big(f - b\varphi\big)(x) \ge 0\} \quad \text{(measurable set)}.$$

$$f_j(x) \le f_{j+1}(x)\ \forall x,\ \forall j \ \Rightarrow\ E_j \subset E_{j+1}\ \forall j.$$

Claim: $\mathbb{R}^n = \bigcup_{j=1}^\infty E_j$.
Let $x \in \mathbb{R}^n$ be arbitrary.

$$\text{If } \varphi(x) = 0, \text{ then } x \in E_1.$$

$$\begin{gathered} \text{If } \varphi(x) > 0 \text{ then } b\varphi(x) < \varphi(x) \le f(x) \quad \text{(because } 0 < b < 1 \text{ and } \varphi(x) < \infty\text{)}. \\ \Rightarrow\ \exists j \text{ s.t. } b\varphi(x) \le f_j(x) \ \Rightarrow\ x \in E_j. \\ \text{Hence } \mathbb{R}^n = \bigcup_{j=1}^\infty E_j. \end{gathered}$$

$$f_j \geq f_j\chi_{E_j} \geq b\varphi\chi_{E_j}$$
$$\Rightarrow \int_{\mathbb{R}^n} f_j \geq \int_{\mathbb{R}^n} b\varphi\chi_{E_j} = bI(\varphi, E_j) \xrightarrow{3.4} bI\big(\varphi, \underbrace{\bigcup_{j=1}^{\infty} E_j}_{=\mathbb{R}^n}\big) = bI(\varphi), \text{ as } j \to \infty$$
$$\Rightarrow a = \lim_{j\to\infty} \int_E f_j \geq bI(\varphi) \quad \forall \varphi \in Y, \ \varphi \leq f$$
$$\overset{\text{sup}}{\Longrightarrow} a \geq b\int_{\mathbb{R}^n} f \quad \forall\ 0 < b < 1$$
$$\overset{b\to 1-}{\Longrightarrow} a \geq \int_{\mathbb{R}^n} f.$$

□

Remark. The order of $\int$ and lim can not be changed in general: Example:

$$f_j = j\chi_{(0,1/j]}, \quad f_j \in Y, \quad I(f_j) = j\frac{1}{j} = 1 \quad \forall\ j$$
$$f_j(x) \xrightarrow{j\to\infty} 0 \quad \forall\ x \in \mathbb{R}$$
$$\Rightarrow \int_{\mathbb{R}} \lim_{j\to\infty} f_j = 0 \neq 1 = \lim_{j\to\infty} \int_{\mathbb{R}} f_j \quad \text{(the sequence } (f_j) \text{ is not increasing)}.$$

Example. Find the limit
$$\lim_{x\to 0+} \int_0^{\infty} \frac{e^{-xt}}{1+t^2}\, dt.$$

<u>Solution:</u> It's enough to study the limit
$$\lim_{n\to\infty} \int_0^{\infty} \frac{e^{-x_n t}}{1+t^2}\, dt$$

<u>for all</u> sequences (x_n) s.t. $x_n \geq x_{n+1} > 0$ and $x_n \searrow 0$. Denote
$$f_n(t) = \frac{e^{-x_n t}}{1+t^2}, \quad t \in [0,\infty) \text{ and } n = 1, 2, \ldots$$

$$x_n \geq x_{n+1} > 0 \text{ and } t \in [0,\infty) \ \Rightarrow\ e^{-x_n t} \leq e^{-x_{n+1} t}$$
$$\Rightarrow\ 0 \leq f_n(t) = \frac{e^{-x_n t}}{1+t^2} \leq \frac{e^{-x_{n+1} t}}{1+t^2} = f_{n+1}(t),$$

that is, the sequence (f_n) is increasing. Furthermore,
$$f_n(t) = \frac{e^{-x_n t}}{1+t^2} \xrightarrow{n\to\infty} \frac{e^{0\cdot t}}{1+t^2} = \frac{1}{1+t^2} \quad \forall\ t \in [0,\infty).$$

MCT $\Rightarrow$
$$\lim_{n\to\infty} \int_0^{\infty} f_n(t)\, dt = \int_0^{\infty} \lim_{n\to\infty} f_n(t)\, dt = \int_0^{\infty} \frac{1}{1+t^2}\, dt \overset{(*)}{=} \lim_{j\to\infty} \int_0^{j} \frac{1}{1+t^2}\, dt$$
$$\overset{3.13}{=} \lim_{j\to\infty} \Big/_0^j \arctan t = \lim_{j\to\infty} \big(\arctan j - \arctan 0\big) = \pi/2.$$

Reason for (∗): MCT applied to the increasing sequence (g_j),

$$g_j(t) = \frac{\chi_{[0,j]}(t)}{1+t^2}.$$

(Note: In Theorem 3.13 the set E is bounded.)

Theorem 3.18. *Let $E \subset \mathbb{R}^n$ be measurable and $f_1, \dots, f_k \colon E \to \dot{\mathbb{R}}$ measurable s.t. $f_j \geq 0$. Then*

$$\int_E \sum_{j=1}^k f_k = \sum_{j=1}^k \int_E f_k.$$

Proof. We may assume: $E = \mathbb{R}^n$ and $k = 2$. Theorem 3.9 ⇒ ∃ increasing sequences (φ_j), (ψ_j) of simple functions s.t.

$$\varphi_j \nearrow f_1 \quad \text{and} \quad \psi_j \nearrow f_2 \quad \text{as } j \to \infty.$$

$$\left.\begin{array}{r} 3.5 \;\Rightarrow\; I(\varphi_j + \psi_j) = I(\varphi_j) + I(\psi_j) \\[1em] \text{MCT} \;\Rightarrow\; I(\varphi_j) = \int \varphi_j \to \int f_1 \;\text{ and }\; I(\psi_j) \to \int f_2, \\[1em] \text{similarly, } \varphi_j + \psi_j \nearrow f_1 + f_2 \text{ and MCT} \;\Rightarrow \\[1em] I(\varphi_j + \psi_j) \to \int (f_1 + f_2) \end{array}\right\} \Rightarrow \int (f_1 + f_2) = \int f_1 + \int f_2 .$$

□

Beppo Levi Theorem.

Theorem 3.19. *Let $E \subset \mathbb{R}^n$ be measurable and $f_j \colon E \to \dot{\mathbb{R}}$ measurable s.t. $f_j \geq 0$. Then*

$$\int_E \Big(\sum_{j\in\mathbb{N}} f_j\Big) = \sum_{j\in\mathbb{N}} \int_E f_j.$$

Proof. Denote $u_k = \sum_{j=1}^k f_j$. Then

$$0 \leq u_1 \leq u_2 \leq \cdots \quad \text{and} \quad u_k \to \sum_{j=1}^\infty f_j =: u.$$

MCT and Thm. 3.18 ⇒

$$\int_E u = \int_E \lim_{k\to\infty} u_k \overset{\text{MCT}}{=} \lim_{k\to\infty} \int_E u_k \overset{3.18}{=} \lim_{k\to\infty} \sum_{j=1}^k \int_E f_j = \sum_{j=1}^\infty \int_E f_j.$$

□

The next convergence result is also very important!

Theorem 3.20. (**Fatou's lemma**). *Let $E \subset \mathbb{R}^n$ be measurable and $f_j \colon E \to \dot{\mathbb{R}}$ measurable s.t. $f_j \geq 0 \;\forall\; j \in \mathbb{N}$. Then*

$$\int_E \liminf_{j\to\infty} f_j \leq \liminf_{j\to\infty} \int_E f_j \qquad (+\infty \textit{ allowed}).$$

Proof. Denote

$$g_k(x) = \inf_{j \geq k} f_j(x), \quad x \in E.$$

Then

$$0 \leq g_k \leq g_{k+1} \quad \forall\, k \in \mathbb{N}$$
$$g_k \ \text{ measurable (Thm. 2.14)}$$
$$g_k \leq f_k \ \text{ and } \ \lim_{k\to\infty} g_k = \liminf_{j\to\infty} f_j$$

$$\text{MCT} \ \Rightarrow \ \int_E \liminf_{j\to\infty} f_j = \int_E \lim_{k\to\infty} g_k \overset{\text{MCT}}{=} \lim_{k\to\infty} \int_E g_k = \liminf_{k\to\infty} \int_E g_k \underset{g_k \leq f_k}{\leq} \liminf_{k\to\infty} \int_E f_k\,.$$

□

Example. (1)

$$f_j = j\chi_{(0,1/j]}$$
$$\lim_{j\to\infty} f_j(x) = 0 \ \forall\, x \in \mathbb{R} \ \Rightarrow \ \liminf_{j\to\infty} f_j = 0$$
$$\int_{\mathbb{R}} f_j = 1 \ \forall\, j$$

Fatou's lemma holds in the form $0 \leq 1$.

(2)

$$f_j = \chi_{[j,2j]}$$
$$\lim_{j\to\infty} f_j(x) = 0 \ \forall\, x \in \mathbb{R} \ \Rightarrow \ \liminf_{j\to\infty} f_j = 0$$
$$\int_{\mathbb{R}} f_j = m([j,2j]) = j \to \infty \ \text{ as } j \to \infty$$

Fatou's lemma holds in the form $0 \leq \infty$.

Integral as a set function is a measure:

Theorem 3.21. *Let $f\colon \mathbb{R}^n \to \dot{\mathbb{R}}$ be measurable, $f \geq 0$. Then the mapping*

$$\operatorname{Leb}\mathbb{R}^n \to [0,+\infty], \quad E \mapsto \int_E f$$

is a <u>*measure*</u>*, i.e.*

(i)

$$\int_\emptyset f = 0\,,$$

(ii) if $E_j \subset \mathbb{R}^n$ are measurable and disjoint, then

$$\int_{\bigcup_{j=1}^\infty E_j} f = \sum_{j=1}^\infty \int_{E_j} f\,.$$

In particular,

(iii) $E_1 \subset E_2 \subset \cdots \subset \mathbb{R}^n$ *measurable* $\Rightarrow$

$$\int_{\bigcup_{j=1}^{\infty} E_j} f = \lim_{j\to\infty} \int_{E_j} f\,,$$

(iv) $\mathbb{R}^n \supset E_1 \supset E_2 \supset \cdots$ *measurable and* $\int_{E_1} f < \infty \;\Rightarrow$

$$\int_{\bigcap_{j=1}^{\infty} E_j} f = \lim_{j\to\infty} \int_{E_j} f\,,$$

Proof. (i): Thm. 3.12 (4); (ii): Exerc.; (iii) and (iv): Theorems on convergence of measures 1.32 and 1.33. □

Theorem 3.22. *(i) Let* $f, g\colon E \to \dot{\mathbb{R}}$ *be measurable and* $f \geq 0,\; g \geq 0$. *If* $f = g$ *a.e. in* E*, then*

$$\int_E f = \int_E g\,.$$

In particular: $f \geq 0$ *measurable and defined a.e. in* $E \;\Rightarrow\; \int_E f$ *well-defined.*

(ii) Let $f\colon E \to \dot{\mathbb{R}}$ *be measurable,* $f \geq 0$. *If* $\int_E f = 0$*, then* $f = 0$ *a.e. in* E.

Proof. (i): Denote $A = \{x \in E\colon f(x) \neq g(x)\}$. By assumption $m(A) = 0$.

$$\int_E f \overset{3.21}{=} \int_{\underbrace{E\setminus A}_{f=g}} f + \underbrace{\int_A f}_{=0} = \int_{E\setminus A} g + \int_A g = \int_E g.$$

(ii): Assume on the contrary that $m\big(\{x \in E\colon f(x) > 0\}\big) > 0$. By Exercise, $\exists\, r > 0$ s.t.

$$m\big(\underbrace{\{x \in E\colon f(x) > r\}}_{\text{denote } =A}\big) > 0$$

$$\Rightarrow \int_E f \overset{(*)}{\geq} \int_A f \overset{(**)}{\geq} r\int_A \chi_A = r m(A) > 0. \quad \underline{\text{contradiction}}$$

$$[(*):\; A \subset E, \qquad (**):\; f\chi_A \geq r\chi_A]$$

□

Remark: Let (X, Γ, μ) be a measure space, f Γ-measurable function $X \to [0,\infty]$. Define the <u>integral</u> of f

$$\int_X f = \sup\{I(\varphi)\colon \varphi\colon X \to \mathbb{R} \text{ simple},\; \varphi \leq f\},$$

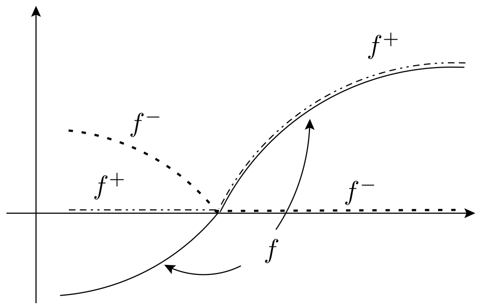
$$\int_E f = \int_X f\chi_E \quad \text{if } E \in \Gamma.$$

The results in Section 3.8 (except Theorem 3.13 (Riemann int.)) hold.

3.23 Lebesgue integral: general case

Let $f\colon E \to \dot{\mathbb{R}}$ be measurable and $E \subset \mathbb{R}^n$. Denote

$$f^+(x) = \max\{f(x), 0\} \qquad (= \tfrac{1}{2}\big(|f| + f\big) \ \text{measurable})$$
$$f^-(x) = -\min\{f(x), 0\} \qquad (= \tfrac{1}{2}\big(|f| - f\big) \ \text{measurable}).$$



Then

$$f^+(x) \geq 0, \qquad f^-(x) \geq 0$$

$$f(x) = f^+(x) - f^-(x), \qquad |f(x)| = f^+(x) + f^-(x).$$

(Note: above the case $\infty - \infty$ does not occur because either $f^+(x) = 0$ or $f^-(x) = 0$.)
Section 3.8 $\Rightarrow$

$$\int_E f^+ \quad \text{and} \quad \int_E f^- \quad \text{defined} \ \ (\in [0, +\infty]).$$

Can we always define

$$\int_E f = \int_E f^+ - \int_E f^- \quad (\text{cf. } f = f^+ - f^-)?$$

No(!) since now the (undefined) case $\infty - \infty$ may occur!

Definition. A function $f\colon E \to \dot{\mathbb{R}}$ is integrable in E if f is measurable and $\int_E f^+ < \infty$ and $\int_E f^- < \infty$. Then the integral of f over E is

$$\int_E f = \int_E f^+ - \int_E f^- \quad (\in \mathbb{R}).$$

Theorem 3.24. *A function $f\colon E \to \dot{\mathbb{R}}$ is integrable in E* $\iff$ *f measurable and*

$$\int_E |f| < \infty.$$

Then

$$\Big|\int_E f\Big| \leq \int_E |f|.$$

Proof. $\boxed{\Rightarrow}$ Measurability is included in the definition of integrability. Furthermore,

$$|f| = \underbrace{f^+}_{\geq 0} + \underbrace{f^-}_{\geq 0} \overset{3.18}{\Longrightarrow} \int_E |f| = \underbrace{\int_E f^+}_{<\infty} + \underbrace{\int_E f^-}_{<\infty} < \infty.$$

$\boxed{\Leftarrow}$

$$\left.\begin{aligned} 0 \leq f^+ \leq |f| \ &\Rightarrow\ \textstyle\int_E f^+ \leq \int_E |f| < \infty \\ 0 \leq f^- \leq |f| \ &\Rightarrow\ \textstyle\int_E f^- \leq \int_E |f| < \infty \end{aligned}\right\} \quad \Rightarrow \quad f \text{ integrable in } E.$$

Furthermore,

$$\begin{aligned} \left|\int_E f\right| &= \left|\int_E f^+ - \int_E f^-\right| \leq \underbrace{\left|\int_E f^+\right|}_{\geq 0} + \underbrace{\left|\int_E f^-\right|}_{\geq 0} = \int_E f^+ + \int_E f^- \\ &\overset{3.18}{=} \int_E (f^+ + f^-) = \int_E |f|. \end{aligned}$$

□

Remark. f integrable in $E \overset{3.16,\ 3.24}{\Longrightarrow} |f(x)| < \infty$ a.e. $x \in E$.

Theorem 3.25. *If $f\colon E \to \dot{\mathbb{R}}$ is measurable, $|f| \leq g$ and g integrable in E, then f is integrable in E.*

Proof.

$$\int_E |f| \leq \int_E g < \infty.$$

□

Remark. It suffices that $|f| \leq g$ a.e. in E, i.e.

$$m\big(\underbrace{\{x \in E\colon |f(x)| > g(x)\}}_{=A}\big) = 0, \quad \text{then} \quad \int_E |f| = \underbrace{\int_{E\setminus A} |f|}_{<\infty} + \underbrace{\int_A |f|}_{=0} < \infty.$$

Theorem 3.26. *If $f\colon E \to \mathbb{R}$ is measurable and Riemann integrable, then f is Lebesgue integrable in E and*

$$\int_E f = (\mathrm{R}) \int_E f\,.$$

Proof.

$$\begin{aligned} &f^+ = \frac{1}{2}\big(|f| + f\big), \quad f^- = \frac{1}{2}\big(|f| - f\big) \quad \text{Riemann integrable} \\ &\overset{3.13}{\Longrightarrow} f^+ \text{ ja } f^+ \text{ Leb. integrable and Riem./Leb.-integrals are same} \\ &\Rightarrow \int_E f = \int_E f^+ - \int_E f^- = (\mathrm{R}) \int_E f^+ - (\mathrm{R}) \int_E f^- = (\mathrm{R}) \int_E f\,. \end{aligned}$$

□

Theorem 3.27. *Let $E \subset \mathbb{R}^n$ be measurable, $f, g\colon E \to \dot{\mathbb{R}}$ integrable in E and $\lambda \in \mathbb{R}$. Then* *(i)* $f+g$ *integrable in* E *and* $\int_E (f+g) = \int_E f + \int_E g$;

(ii) λf *integrable in* E *and* $\int_E \lambda f = \lambda \int_E f$;

(iii) $f \le g \;\Rightarrow\; \int_E f \le \int_E g$;

(iv) $m(E) = 0 \;\Rightarrow\; \int_E f = 0$;

(v) $f = g$ *a.e. in* $E \;\Rightarrow\; \int_E f = \int_E g$.

Remark. f, g integrable in $E \;\Rightarrow\; f(x), g(x) \in \mathbb{R}$ a.e. $x \in E \;\Rightarrow\; f+g$ defined a.e. in E.

Proof. (i): Let $h = f + g$. Then h defined a.e. and measurable

$$|h| \le |f| + |h| \;\Rightarrow\; \int_E |h| \le \int_E |f| + \int_E |g| < \infty \;\Rightarrow\; h \text{ integrable}$$

In general, $h^+ \ne f^+ + g^+$, but a.e. in E:

$$h^+ - h^- = h = f + g = f^+ - f^- + g^+ - g^-$$

$$\Rightarrow\; h^+ + f^- + g^- = h^- + f^+ + g^+ \qquad \text{(functions} \ge 0\text{, integrate both sides (Thm. 3.18))}$$

$$\Rightarrow\; \int_E h^+ + \int_E f^- + \int_E g^- = \int_E h^- + \int_E f^+ + \int_E g^+ \qquad \text{(integraalit} < \infty\text{)}$$

$$\Rightarrow\; \int_E h = \int_E h^+ - \int_E h^- = \int_E f^+ - \int_E f^- + \int_E g^+ - \int_E g^-$$
$$= \int_E f + \int_E g.$$

(ii): (a) $\lambda \ge 0$

$$(\lambda f)^+ = \lambda f^+ \quad \text{ja} \quad (\lambda f)^- = \lambda f^-$$
$$\Rightarrow\; \int_E (\lambda f)^+ = \lambda \int_E f^+ \quad \text{ja} \quad \int_E (\lambda f)^- = \lambda \int_E f^-$$
$\Rightarrow$ claim

(b) $\lambda < 0$

$$(\lambda f)^+ = (-\lambda) f^- \quad \text{ja} \quad (\lambda f)^- = (-\lambda) f^+, \quad \text{and the claim follows as above}$$

(iii): (i) and (ii) $\Rightarrow$ $g - f$ integrable and

$$\int_E g = \int_E f + \int_E \underbrace{(g-f)}_{\ge 0} \ge \int_E f$$

(iv): $m(E) = 0 \;\Rightarrow\; \int_E f^+ = 0$ and $\int_E f^- = 0 \quad \Rightarrow\; \int_E f = 0$

(v): $f = g$ a.e. in $E \;\Rightarrow\; f^+ = g^+,\; f^- = g^-$ a.e. in E

$$\Rightarrow\; \int_E f^+ = \int_E g^+ \quad \text{ja} \quad \int_E f^- = \int_E g^- \;\Rightarrow\; \text{claim.}$$

$\square$

Convergence theorems

Theorem 3.28. (**Dominated convergence theorem, DCT**) *Let* $E \subset \mathbb{R}^n$ *be measurable and* (f_j), $j \in \mathbb{N}$, *a sequence of measurable functions s.t.*

$$f(x) = \lim_{j\to\infty} f_j(x) \quad a.e.\ x \in E\,.$$

If $\exists\ g\colon E \to \dot{\mathbb{R}}$ *s.t.* g *is integrable in* E *and*

$$|f_j(x)| \le g(x),\ \forall\ j \in \mathbb{N},\ \text{and a.e. } x \in E\,,$$

then f *is integrable in* E *and*

$$\int_E f = \lim_{j\to\infty} \int_E f_j\,. \qquad (\text{Note } \int_E f \in \mathbb{R})$$

Proof. By redefining f_j, f and g in a set of measure zero, we may assume

$$\begin{aligned} &f_j(x) \xrightarrow{j\to\infty} f(x) \quad \forall\ x \in E \ \text{ and} \\ &|f_j(x)| \le g(x) \quad \forall\ x \in E \end{aligned}$$

$$\Rightarrow |f(x)| \le |g(x)| \quad \forall\ x \in E\,.$$

g integrable in E, Thm. 3.25) $\Rightarrow$ f integrable in E.

$$g + f_j \ge 0 \quad \text{and} \quad g + f_j \to g + f \overset{\text{Fatou}}{\Longrightarrow}$$

$$\begin{aligned} \int_E g + \int_E f = \int_E (g+f) &\overset{\text{Fatou}}{\le} \liminf_{j\to\infty} \int_E (g + f_j) = \liminf_{j\to\infty} \Big(\int_E g + \int_E f_j \Big) \\ &= \int_E g + \liminf_{j\to\infty} \int_E f_j \end{aligned}$$

$$\Rightarrow \int_E f \le \liminf_{j\to\infty} \int_E f_j \quad (\text{note } \int_E g < \infty)$$

$$g - f_j \ge 0\,, \quad \text{therefore}$$

$$\begin{aligned} \int_E g - \int_E f = \int_E (g-f) &\overset{\text{Fatou}}{\le} \liminf_{j\to\infty} \int_E (g - f_j) = \liminf_{j\to\infty} \Big(\int_E g - \int_E f_j \Big) \\ &= \int_E g - \limsup_{j\to\infty} \int_E f_j \end{aligned}$$

$$\Rightarrow \int_E f \ge \limsup_{j\to\infty} \int_E f_j\,.$$

Hence

$$\int_E f \le \liminf_{j\to\infty} \int_E f_j \le \limsup_{j\to\infty} \int_E f_j \le \int_E f \quad \Rightarrow \quad \text{claim} \ \square$$

$\square$

Example. Find the limit

$$\lim_{n\to\infty} n\int_0^1 x^{-3/2}\sin\frac{x}{n}\,dx\,.$$

Let $f_n(x) = nx^{-3/2}\sin\frac{x}{n} = \big(\underbrace{(n/x)\sin(x/n)}_{\to 1,\text{ as } n\to\infty}\big)x^{-1/2} \xrightarrow{n\to\infty} x^{-1/2} \stackrel{\text{def.}}{=} f(x)$, then

$$\int_0^1 f = \Big/_0^1 2\sqrt{x} = 2.$$

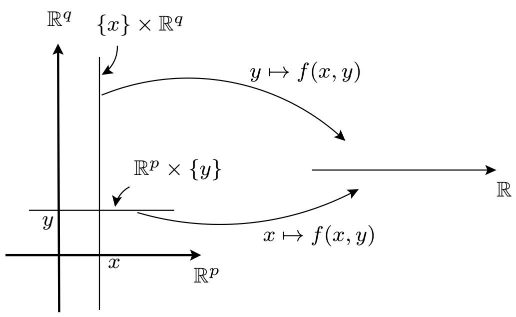
$$\begin{aligned}|\sin t| \le t\ \forall\ t\ge 0 \ \Rightarrow\ |(n/x)\sin(x/n)| \le 1 \quad \forall\ n\in\mathbb{N},\ \forall\ x\in(0,1]\\ \Rightarrow\ |f_n(x)| \le x^{-1/2} = g(x)\ \big(= f(x)\big),\ g \text{ integrable in } [0,1]\\ \text{DCT}\ \Rightarrow\ \int_0^1 f_n \to \int_0^1 f = 2.\end{aligned}$$

4 Fubini's theorems

Here we just present Fubini's theorems without proofs.

We identify $\mathbb{R}^{p+q} = \mathbb{R}^p\times\mathbb{R}^q,\ p,q\in\mathbb{N}$.

$$z\in\mathbb{R}^{p+q} \iff z = (\underbrace{x_1,\dots,x_p}_{=x\in\mathbb{R}^p},\underbrace{y_1,\dots,y_q}_{=y\in\mathbb{R}^q}) = (x,y).$$



Theorem 4.1. (**Fubini's 1. theorem,** $f\ge 0$) *Let $f\colon\mathbb{R}^{p+q}\to\dot{\mathbb{R}}$ be measurable and $f\ge 0$. Then*

(1)

$$y\mapsto f(x,y)\ \textit{is measurable for a.e. } x\in\mathbb{R}^p\,;$$
$$[\text{i.e. } m_p\big(\{x\in\mathbb{R}^p\colon y\mapsto f(x,y)\text{ non-measurable}\}\big)=0]$$

(2)

$$x\mapsto f(x,y)\ \textit{is measurable for a.e. } y\in\mathbb{R}^q\,;$$

(3)

$$x\mapsto\int_{\mathbb{R}^q} f(x,y)\,dm_q(y)\quad\textit{is measurable;}$$

(4)

$$y \mapsto \int_{\mathbb{R}^p} f(x,y)\, dm_p(x) \quad \textit{measurable;}$$

(5)

$$\int_{\mathbb{R}^{p+q}} f = \int_{\mathbb{R}^p} \left(\int_{\mathbb{R}^q} f(x,y)\, dm_q(y) \right) dm_p(x)$$
$$= \int_{\mathbb{R}^q} \left(\int_{\mathbb{R}^p} f(x,y)\, dm_p(x) \right) dm_q(y)\,. \quad (+\infty \text{ allowed})$$

Theorem 4.2. (**Fubini's 2. theorem**, general case) *Let $f\colon \mathbb{R}^{p+q} \to \dot{\mathbb{R}}$ be measurable and suppose that at least one of the integrals*

$$\int_{\mathbb{R}^{p+q}} |f|\,, \quad \int_{\mathbb{R}^p} \left(\int_{\mathbb{R}^q} |f(x,y)|\, dm_q(y) \right) dm_p(x)\,, \quad \textit{or}$$

$$\int_{\mathbb{R}^q} \left(\int_{\mathbb{R}^p} |f(x,y)|\, dm_p(x) \right) dm_q(y)$$

is finite. Then

(1) *$y \mapsto f(x,y)$ is integrable over $\mathbb{R}^q$ for a.e. $x \in \mathbb{R}^p$;*

(2) *$x \mapsto f(x,y)$ is integrable over $\mathbb{R}^p$ for a.e. $y \in \mathbb{R}^q$;*

(3) *$x \mapsto \int_{\mathbb{R}^q} f(x,y)\, dm_q(y)$ is integrable over $\mathbb{R}^p$, i.e.*

$$\int_{\mathbb{R}^p} \left| \int_{\mathbb{R}^q} |f(x,y)|\, dm_q(y) \right| dm_p(x) < \infty\,;$$

(4) *$y \mapsto \int_{\mathbb{R}^p} f(x,y)\, dm_p(x)$ is integrable over $\mathbb{R}^q$;*

(5) *f is integrable over $\mathbb{R}^{p+q}$, and*

$$\int_{\mathbb{R}^{p+q}} f = \int_{\mathbb{R}^p} \left(\int_{\mathbb{R}^q} f(x,y)\, dm_q(y) \right) dm_p(x) = \int_{\mathbb{R}^q} \left(\int_{\mathbb{R}^p} f(x,y)\, dm_p(x) \right) dm_q(y)\,. \quad (\in \mathbb{R})$$

Below is a list of (some) books that can be used as an additional material.

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