BRIEF INTRODUCTION TO FOURIER SERIES AND HAAR WAVELETS

Version 3, Samuli Siltanen, March 1, 2017

1. Fourier series, real formulation

Assume that the function $f : \mathbb{R} \to \mathbb{R}$ is 2π-periodic (in other words, satisfies $f(x) = f(x + \nu 2\pi)$ for any $\nu \in \mathbb{Z}$) and can be written in the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right), \tag{1}$$

where $a_0, a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ are real-valued coefficients.

Computationally it is very useful to consider approximations of functions and signals by truncated Fourier series

$$f(x) \approx a_0 + \sum_{n=1}^{N} \left(a_n \cos(nx) + b_n \sin(nx) \right). \tag{2}$$

Then the practical question is: given f, how to determine the coefficients $a_0, a_1, a_2, \ldots, a_N$ and $b_1, b_2, \ldots, b_N$? Let us derive formulas for them.

The constant coefficient a_0 is found as follows. Integrate both sides of (1) from 0 to 2π:

$$\begin{aligned} \int_0^{2\pi} f(x)dx &= a_0 \int_0^{2\pi} dx + \\ &\quad + \sum_{n=1}^{\infty} a_n \int_0^{2\pi} \cos(nx)dx + \\ &\quad + \sum_{n=1}^{\infty} b_n \int_0^{2\pi} \sin(nx)dx, \end{aligned} \tag{3}$$

where we assumed that the orders of infinite summing and integration can be interchanged. Now it is easy to check that $\int_0^{2\pi} \cos(nx)dx = 0$ and $\int_0^{2\pi} \sin(nx)dx = 0$ and $\int_0^{2\pi} dx = 2\pi$. Therefore,

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)dx, \tag{4}$$

which can be interpreted as the average value of the function f over the interval $[0, 2\pi]$.

Further, fix any integer $m \geq 1$ and multiply both sides of (1) by $\cos(mx)$. Integration from 0 to 2π gives

$$\begin{aligned}\int_0^{2\pi} f(x)\cos(mx)dx &= a_0 \int_0^{2\pi} \cos(mx)dx + \\ &+ \sum_{n=1}^{\infty} a_n \int_0^{2\pi} \cos(nx)\cos(mx)dx + \\ &+ \sum_{n=1}^{\infty} b_n \int_0^{2\pi} \sin(nx)\cos(mx)dx. \end{aligned} \tag{5}$$

We already know that $\int_0^{2\pi} \cos(mx)dx = 0$, so the term containing a_0 in the right hand side of (5) vanishes. Clever use of trigonometric identities allows one to see that

$$\int_0^{2\pi} \sin(nx)\cos(mx)dx = 0 \quad \text{for all } n \geq 1, \tag{6}$$

and that

$$\int_0^{2\pi} \cos(nx)\cos(mx)dx = 0 \quad \text{for all } n \geq 1 \text{ with } n \neq m. \tag{7}$$

The checking of (6) and (7) is left as an exercise. So actually the only nonzero term in the right hand side of (5) is the one containing the coefficient a_m. Another exercise is to verify this identity:

$$\int_0^{2\pi} \cos(nx)\cos(nx)dx = \pi. \tag{8}$$

Therefore, substituting (8) into (5) gives

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)dx. \tag{9}$$

A similar derivation shows that

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)dx. \tag{10}$$

One might be tempted to ask: what kind of functions allow a representation of the form (1)? Or: in what sense does the right-hand sum converge in (2) as $N \to \infty$? Also: under what assumptions can the order of infinite summing and integration can be interchanged in the derivations of (3) and (5)? These are deep and interesting mathematical questions which will not be further discussed in this short note.

2. Fourier series, complex formulation

Parametrize the boundary of the unit circle as

$$\{(\cos\theta, \sin\theta) \,|\, 0 \le \theta < 2\pi\}.$$

We will use the *Fourier basis functions*

$$\varphi_n(\theta) = (2\pi)^{-1/2} e^{in\theta}, \quad n \in \mathbb{Z}. \tag{11}$$

We can approximate 2π-periodic functions $f : \mathbb{R} \to \mathbb{R}$ following the lead of the great applied mathematician Joseph Fourier (1768–1830). Define cosine series coefficients using the L^2 inner product

$$\widehat{f}_n := \langle f, \varphi_n \rangle = \int_0^{2\pi} f(\theta)\, \overline{\varphi_n(\theta)}\, d\theta, \qquad n \in \mathbb{Z}.$$

Then, for nice enough functions f, we have

$$f(\theta) \approx \sum_{n=-N}^{N} \widehat{f}_n\, \varphi_n(\theta)$$

with the approximation getting better when N grows.

Note that the functions φ_n are orthogonal:

$$\langle \varphi_n, \varphi_n \rangle = \delta_{nm}.$$

3. Haar wavelets

For a wonderful introduction to wavelets, please see the classic book *Ten lectures on wavelets* by Ingrid Daubechies [1].

3.1. Theoretical approach as orthonormal basis of $L^2([0,1])$. Consider real-valued functions defined on the interval $[0,1]$. There are two especially important functions, namely the *scaling function* $\varphi(x)$ and the *mother wavelet* $\psi(x)$ related to the Haar wavelet basis, defined as follows:

$$\varphi(x) \equiv 1, \qquad \psi(x) = \begin{cases} 1, & \text{for } 0 \le x < 1/2, \\ -1 & \text{for } 1/2 \le x \le 1. \end{cases}$$

Also, let us define *wavelets* as scaled and translated versions of the mother wavelet:

$$\psi_{jk} := 2^{j/2}\psi(2^j x - k) \qquad \text{for } j \le 0 \text{ and } 0 \le k \le 2^j - 1.$$

Let $f, g : [0,1] \to \mathbb{R}$. Define the inner product between f and g by

$$\langle f, g \rangle := \int_0^1 f(x)\overline{g(x)}\, dx. \tag{12}$$

(Note that the complex conjugate over g in (12) is not relevant here as g is real-valued. We just have it there for mathematical completeness.) Please convince yourself about the fact that wavelets are orthogonal:

$$\langle \psi_{jk}, \psi_{j'k'} \rangle = \begin{cases} 1 & \text{if } j = j' \text{ and } k = k', \\ 0 & \text{otherwise.} \end{cases}$$

(Start by understanding why $\langle \psi, \varphi \rangle = 0$ and $\langle \psi, \psi \rangle = 1$, then look at smaller scales corresponding to $j > 0$. Basically it is the same phenomenon always.)

3.2. **Computational implementation of Haar wavelet transform.** Consider a signal $\mathbf{f} \in \mathbb{R}^n$, where $n = 2^{n_0}$ with some $n_0 > 1$. Think of $\mathbf{f}$ as a collection of point values of a function $f : [0,1] \to \mathbb{R}$ evaluated at an equispaced grid. For simplicity, let us explain the whole process first for $\mathbf{f} \in \mathbb{R}^4$ and then in general.

3.2.1. *One-step non-normalized Haar transform.* Consider two convolution operations on $\mathbf{f} \in \mathbb{R}^4$. The *low-pass filter* involves convolution with the filter $\mathbf{g}_0 = [1\ 1]^T \in \mathbb{R}^2$, and the *detail filter* convolves with $\mathbf{d}_0 = [1\ -1]^T \in \mathbb{R}^2$. Then

$$\mathbf{g}_0 * \mathbf{f} = \begin{bmatrix} \mathbf{f}_1 + \mathbf{f}_2 \\ \mathbf{f}_2 + \mathbf{f}_3 \\ \mathbf{f}_3 + \mathbf{f}_4 \\ \mathbf{f}_4 + 0 \end{bmatrix}, \qquad \mathbf{d}_0 * \mathbf{f} = \begin{bmatrix} \mathbf{f}_2 - \mathbf{f}_1 \\ \mathbf{f}_3 - \mathbf{f}_2 \\ \mathbf{f}_4 - \mathbf{f}_3 \\ 0 - \mathbf{f}_4 \end{bmatrix},$$

where we used zero boundary conditions for the convolution.

The next step is to apply a *downsampling* operation. Denote by DS_2 the operator that picks out every other component of a vector: $\mathrm{DS}_2([\mathbf{f}_1\, \mathbf{f}_2\, \mathbf{f}_3\, \mathbf{f}_4]) = [\mathbf{f}_1\, \mathbf{f}_3]$. Then

$$\mathrm{DS}_2(\mathbf{g}_0 * \mathbf{f}) = \begin{bmatrix} \mathbf{f}_1 + \mathbf{f}_2 \\ \mathbf{f}_3 + \mathbf{f}_4 \end{bmatrix}, \qquad \mathrm{DS}_2(\mathbf{d}_0 * \mathbf{f}) = \begin{bmatrix} \mathbf{f}_2 - \mathbf{f}_1 \\ \mathbf{f}_4 - \mathbf{f}_3 \end{bmatrix}.$$

Now we can define the non-normalized Haar transform of $\mathbf{f} \in \mathbb{R}^4$:

$$\mathrm{Haar}_0(\mathbf{f}) := \left[\frac{\mathrm{DS}_2(\mathbf{g}_0 * \mathbf{f})}{\mathrm{DS}_2(\mathbf{d}_0 * \mathbf{f})} \right] = \begin{bmatrix} \mathbf{f}_1 + \mathbf{f}_2 \\ \mathbf{f}_3 + \mathbf{f}_4 \\ \hline \mathbf{f}_2 - \mathbf{f}_1 \\ \mathbf{f}_4 - \mathbf{f}_3 \end{bmatrix} \in \mathbb{R}^4. \tag{13}$$

3.2.2. *One-step non-normalized inverse Haar transform.* Given a vector $\mathbf{h} = [\mathbf{h}_1\, \mathbf{h}_2\, \mathbf{h}_3\, \mathbf{h}_4]$, the inverse one-step Haar transform starts by picking out the two halves of the vector: $\mathbf{h}^{(1)} = [\mathbf{h}_1\, \mathbf{h}_2]$ and $\mathbf{h}^{(2)} =$

$[\mathbf{h}_3\,\mathbf{h}_4]$. Now apply an *upsampling* operation US_2 that lengthens the two halves by substituting zero elements:

$$\mathrm{US}_2(\mathbf{h}^{(1)}) = \begin{bmatrix} 0 \\ \mathbf{h}_1 \\ 0 \\ \mathbf{h}_2 \end{bmatrix}, \qquad \mathrm{US}_2(\mathbf{h}^{(2)}) = \begin{bmatrix} 0 \\ \mathbf{h}_3 \\ 0 \\ \mathbf{h}_4 \end{bmatrix}.$$

Next we use convolution filters $\widetilde{\mathbf{g}}_0$ and $\widetilde{\mathbf{d}}_0$ defined by reversing the above filters $\mathbf{g}_0$ and $\mathbf{d}_0$. We get $\widetilde{\mathbf{g}}_0 = [1\ 1]^T$ and $\widetilde{\mathbf{d}}_0 = [-1\ 1]^T$, resulting in

$$\widetilde{\mathbf{g}}_0 * \mathrm{US}_2(\mathbf{h}^{(1)}) = \widetilde{\mathbf{g}}_0 * \begin{bmatrix} 0 \\ \mathbf{h}_1 \\ 0 \\ \mathbf{h}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{h}_1 \\ \mathbf{h}_1 \\ \mathbf{h}_2 \\ \mathbf{h}_2 \end{bmatrix}$$

and

$$\widetilde{\mathbf{d}}_0 * \mathrm{US}_2(\mathbf{h}^{(2)}) = \widetilde{\mathbf{d}}_0 * \begin{bmatrix} 0 \\ \mathbf{h}_3 \\ 0 \\ \mathbf{h}_4 \end{bmatrix} = \begin{bmatrix} -\mathbf{h}_3 \\ \mathbf{h}_3 \\ -\mathbf{h}_4 \\ \mathbf{h}_4 \end{bmatrix}.$$

Now we define

$$\mathrm{Haar}_0^{-1}(\mathbf{h}) := \widetilde{\mathbf{g}}_0 * \mathrm{US}_2(\mathbf{h}^{(1)}) + \widetilde{\mathbf{d}}_0 * \mathrm{US}_2(\mathbf{h}^{(2)}) = \begin{bmatrix} \mathbf{h}_1 - \mathbf{h}_3 \\ \mathbf{h}_1 + \mathbf{h}_3 \\ \mathbf{h}_2 - \mathbf{h}_4 \\ \mathbf{h}_2 + \mathbf{h}_4 \end{bmatrix}. \tag{14}$$

3.2.3. *Normalization.* It would be great if Haar_0^{-1} would be the inverse operator of Haar_0. However, this is what we get from combining (13) and (14):

$$\begin{aligned} \mathrm{Haar}_0^{-1}\big(\mathrm{Haar}_0(\mathbf{f})\big) &= \mathrm{Haar}_0^{-1}\left(\left[\begin{array}{c} \mathbf{f}_1 + \mathbf{f}_2 \\ \mathbf{f}_3 + \mathbf{f}_4 \\ \hline \mathbf{f}_2 - \mathbf{f}_1 \\ \mathbf{f}_4 - \mathbf{f}_3 \end{array}\right] \right) \\ &= \begin{bmatrix} \mathbf{f}_1 + \mathbf{f}_2 - (\mathbf{f}_2 - \mathbf{f}_1) \\ \mathbf{f}_1 + \mathbf{f}_2 + \mathbf{f}_2 - \mathbf{f}_1 \\ \mathbf{f}_3 + \mathbf{f}_4 - (\mathbf{f}_4 - \mathbf{f}_3) \\ \mathbf{f}_3 + \mathbf{f}_4 + \mathbf{f}_4 - \mathbf{f}_3 \end{bmatrix} \\ &= \begin{bmatrix} 2\mathbf{f}_1 \\ 2\mathbf{f}_2 \\ 2\mathbf{f}_3 \\ 2\mathbf{f}_4 \end{bmatrix}. \end{aligned}$$

So we have, unfortunately, $\mathrm{Haar}_0^{-1}\big(\mathrm{Haar}_0(\mathbf{f})\big) = 2\mathbf{f}$ instead of just $\mathbf{f}$ as desired.

However, this is easy to fix. Set

$$\mathbf{g} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, \quad \mathbf{d} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}, \quad \widetilde{\mathbf{g}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, \quad \widetilde{\mathbf{d}} = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix},$$

and define

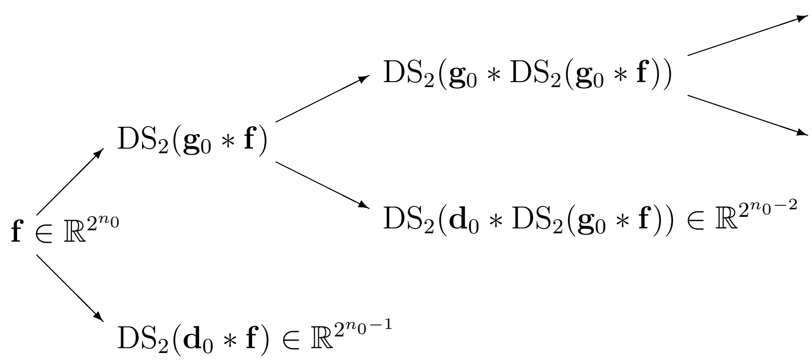
$$\mathrm{Haar}(\mathbf{f}) := \left[\frac{\mathrm{DS}_2(\mathbf{g} * \mathbf{f})}{\mathrm{DS}_2(\mathbf{d} * \mathbf{f})}\right], \tag{15}$$

and

$$\mathrm{Haar}^{-1}(\mathbf{h}) := \widetilde{\mathbf{g}} * \mathrm{US}_2(\mathbf{h}^{(1)}) + \widetilde{\mathbf{d}} * \mathrm{US}_2(\mathbf{h}^{(2)}). \tag{16}$$

Then it is straightforward to check that $\mathrm{Haar}^{-1}\big(\mathrm{Haar}(\mathbf{f})\big) = \mathbf{f}$.

3.2.4. *Multi-step Haar transform.* The idea is to apply the Haar transform recursively to the low-pass filtered part of the signal at each step of the process.



References

[1] I. Daubechies, *Ten lectures on wavelets (Ninth printing, 2006)*, vol. 61 of CBMS-NSF Regional conference series in applied mathematics, SIAM, 2006.