Inverse problems course, Exercise 4 (for the week starting on February 20, 2017)
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Theoretical exercises:

T1. Let us study the problem of finding the vector $\mathbf{f}_0 \in \mathbb{R}^n$ that gives the minimum value for the function $Q : \mathbb{R}^n \to \mathbb{R}$ defined by

$$Q(\mathbf{f}) = \|A\mathbf{f} - \mathbf{m}\|_2^2,$$

where A is a $k\times n$ matrix and $\mathbf{m} \in \mathbb{R}^k$. Assume that $k \geq n$ and that all singular values of A are strictly positive.

(a) Show that Q is continuously differentiable with respect to any $\mathbf{f}_j$ and compute the gradient $\nabla Q(\mathbf{f})$.

(b) Prove that $A^T A$ is invertible. (Hint: this was done in the lecture.)

(c) Set $\nabla Q = 0$ and deduce that the minimizer $\mathbf{f}_0$ is unique and satisfies $\mathbf{f}_0 = (A^T A)^{-1} A^T \mathbf{m}$.

T2. Let $a > 0$. Assume that the Point Spread Function $\psi : \mathbb{R} \to \mathbb{R}$ is infinitely smooth and satisfies $\psi(x) = 0$ for all $|x| \geq a$. Further, assume $\psi(x) \geq 0$ for all $x \in \mathbb{R}$ and that $\int_{-a}^{a} \psi(x)\, dx = 1$.

Define the *Edge Spread Function (ESF)* by the formula

$$\varphi(x) = \int_{-\infty}^{\infty} \psi(y) H(x - y)\, dy,$$

where the *Heaviside function* H is defined by

$$H(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ 1 & \text{for } x \geq 0. \end{cases}$$

Show that

$$\varphi'(x) = \frac{d\varphi}{dx}(x) = \psi(x).$$

Matlab exercises:

M1. **Stacked-form generalized Tikhonov regularization** for the 1D deconvolution problem. Follow the procedure of Problem M2 of Exercise 2. Take a suitable $k = 128$ and simulate discrete convolution data $\widetilde{\mathbf{m}}$ (with a little noise added) using the simulated continuum model. Furthermore, take $n = k$ and let A be the square-shaped measurement matrix from the computational model.

Let us define two discrete differentiation matrices:

$$L = \frac{1}{\Delta x}\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & \cdots & & 0 \\ 0 & -1 & 1 & 0 & 0 & \cdots & & 0 \\ 0 & 0 & -1 & 1 & 0 & \cdots & & 0 \\ \vdots & & & & \ddots & & & \\ \vdots & & & & & \ddots & & \\ 0 & & \cdots & & 0 & -1 & 1 & 0 \\ 0 & & \cdots & & 0 & 0 & -1 & 1 \end{bmatrix}, \tag{1}$$

and

$$L_0 = \frac{1}{\Delta x}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & & 0 \\ -1 & 1 & 0 & 0 & 0 & \cdots & & 0 \\ 0 & -1 & 1 & 0 & 0 & \cdots & & 0 \\ 0 & 0 & -1 & 1 & 0 & \cdots & & 0 \\ \vdots & & & & \ddots & & & \\ \vdots & & & & & \ddots & & \\ 0 & & \cdots & & 0 & -1 & 1 & 0 \\ 0 & & \cdots & & 0 & 0 & -1 & 1 \\ 0 & & \cdots & & 0 & 0 & 0 & -1 \end{bmatrix}. \tag{2}$$

Note that the size of L is $(n-1)\times n$, while the size of L' is $(n+1)\times n$.

(a) Consider the matrix equation $A'\mathbf{f} = \widetilde{\mathbf{m}}'$, where

$$A' = \begin{bmatrix} A \\ \sqrt{\alpha}L \end{bmatrix}, \qquad \widetilde{\mathbf{m}}' = \begin{bmatrix} \widetilde{\mathbf{m}} \\ \mathbf{0} \end{bmatrix}, \tag{3}$$

and I denotes the $n\times n$ identity matrix and $\mathbf{0}$ is a vertical vector with all components equal to zero. Compute reconstruction as

$$\mathbf{f}_\alpha = A' \setminus \widetilde{\mathbf{m}}'. \tag{4}$$

Let the regularization parameter $\alpha > 0$ range over many values. For each α, calculate the relative square norm error between the reconstruction and the Heaviside function. Which value of α gives the smallest error? What seems to be the limit of the reconstruction as $\alpha \to \infty$?

(b) Repeat (a) using L_0 instead of L. What seems to be the limit of the reconstruction as $\alpha \to \infty$?

M2. **From ESF to PSF.** This is a computational version of problem T2 above. Consider the 1D convolution data we collected in the lecture using a camera. As we noticed, the theoretical PSF given by file *PSF.m* is not so accurate. So let us compute an empirical PSF.

(a) Take the low-noise ESF denoted by `m0` in the file *deco03_data_meas.m*. Use the Matlab command `diff` to differentiate the ESF and so to give the PSF. Normalize the PSF and compare the output of the computational model to the measured data. Is the approximation better with the empirical PSF than with the theoretical PSF?

(b) Is the PSF from (a) symmetric? Probably not. Make it symmetric by a command of type `psf = (psf + fliplr(psf))/2` or `psf = (psf + flipud(psf))/2`. Do you get a better fit of the computational model to the measured data?

(c) Compute the PSF from the medium-noisy ESF (called `mn` in the Matlab routine) by regularized numerical differentiation. You can use a matrix of the form

$$A = \begin{bmatrix} \frac{1}{k} & 0 & 0 & 0 & \dots & 0 \\ \frac{1}{k} & \frac{1}{k} & 0 & 0 & \dots & 0 \\ \frac{1}{k} & \frac{1}{k} & \frac{1}{k} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{k} & \frac{1}{k} & \frac{1}{k} & \frac{1}{k} & \dots & \frac{1}{k} \end{bmatrix} \tag{5}$$

and apply truncated SVD. Do you get a PSF close to the one you got in (a)? How about the high-noise ESF?

M3. Use the empirical PSF from Problem M2 and reconstruct the Heaviside function from the three real-data options `m0`, `mn` and `mn2`. Use Tikhonov regularization and the stacked form computation. Is the result better with the difference matrix L or with the identity matrix?