Inverse problems course, Exercise 3 (for the week starting on February 13, 2017)
University of Helsinki
Department of Mathematics and Statistics
Samuli Siltanen, Minh Mach, Santeri Kaupinmäki and Alexander Meaney

Theoretical exercises:

T1. Study the following parts of the textbook:

- Section 3.1: the three conditions of Hadamard for a well-posed problem.
- Figure 3.3: the subspaces related to a matrix understood as a linear map.
- Section 4.1: the Moore-Penrose pseudoinverse.

Consider the measurement model $\mathbf{m} = A\mathbf{f} + \varepsilon$ with A a $k\times n$ matrix. The inverse problem "given $\mathbf{m}$, find $\mathbf{f}$" can be ill-posed as any of the three conditions of Hadamard may fail.

(a) Explain how the Moore-Penrose pseudoinverse takes care of ill-posedness related to Hadamard's conditions on existence and uniqueness.

(b) Why does the Moore-Penrose fail with the stability condition of Hadamard?

Matlab exercises:

M1. **Stacked-form Tikhonov regularization** for the 1D deconvolution problem. Follow the procedure of Problem M2 of Exercise 2. Take a suitable $k = 128$ and simulate discrete convolution data $\widetilde{\mathbf{m}}$ (with a little noise added) using the simulated continuum model. Furthermore, take $n = k$ and let A be the square-shaped measurement matrix from the computational model.

(a) Choose $\alpha = 1$ and consider the matrix equation $A'\mathbf{f} = \widetilde{\mathbf{m}}'$, where

$$A' = \begin{bmatrix} A \\ \sqrt{\alpha} I \end{bmatrix}, \qquad \widetilde{\mathbf{m}}' = \begin{bmatrix} \widetilde{\mathbf{m}} \\ \mathbf{0} \end{bmatrix}, \tag{1}$$

and I denotes the $n\times n$ identity matrix and $\mathbf{0}$ is a vertical n-vector with all components equal to zero. Compute reconstruction $\mathbf{f}_\alpha$ in Matlab using the backslash operator:

$$\mathbf{f}_\alpha = A' \setminus \widetilde{\mathbf{m}}'. \tag{2}$$

Plot original signal and $\mathbf{f}_\alpha$ in the same picture and compare.

(b) Let the regularization parameter $\alpha > 0$ range over many values (for example $\ldots, 10^{-6}, 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}, 10^{0}, 10^{1}, 10^{2}, \ldots$). For each α, calculate the relative square norm error between the reconstruction and the original signal. Which value of α gives the smallest error? What seems to be the limit of the reconstruction as $\alpha \to \infty$?

(c) Repeat (b) using a higher noise level. Is the optimal regularization parameter α smaller or greater than in (b)?

M2. Download *tomo01_RadonMatrix_comp.m* and *tomo02_firstTSVD_comp.m* from the course website. Let the image resolution be 32×32 and choose the number of angles in tomographic measurement to be 63. Construct the 32×32 "true object" **f** using the routine *SqPhantom.m*.

(a) Compute the sinogram using the `radon` command of Matlab. Then compute the "inverse crime" data $\mathbf{m} = A\mathbf{f}$. Test computationally that the sinogram and data are approximately equal. (Note: there are several reshaping operations involved.)

(b) Plot some of the singular vectors, similarly to Figure 4.4 of the textbook.

(c) Find numerically the optimal truncation index for TSVD for the 2D by comparing the reconstruction to ground truth and calculating relative square norm error. (Note: you do not need to compute the TSVD reconstruction for *all* possible truncation indices. Start from a small number and increase until the reconstructions become really bad.)

(d) Repeat (b) for a higher noise level. Is the optimal truncation index smaller or larger than in (b)?

M3. Repeat M2 using only 13 projection angles. What is different in (b)-(d)?

M4. Simulate 2D tomographic data using higher resolution and a matrix-free formulation, thereby avoiding inverse crime. Here "matrix-free" means that you can just use the `radon` command for computing the higher-resolution sinogram; the matrix A is needed only for the low resolution.

You can use the routines on the page
http://wiki.helsinki.fi/display/mathstatHenkilokunta/X-ray+tomography+with+matrices
See the Matlab files there under title *Creating data without inverse crime*.

(a) Construct the 32×32 "true object" **f** using the routine *SqPhantom.m*. Let the number of angles in tomographic measurement to be 63. Compute two sinograms: first simply with the command `radon`, and second with the "inverse crime-free" method using the resolution 64×64, applying `radon`, and downsampling. (This is what's done in the routine *XRMC_NoCrimeData_comp.m* available on the aforementioned webpage.) What is the relative square norm error between the two sinograms? Plot the two sinograms and their difference.

(b) Repeat (a) with the resolutions 128×128 and 256×256. What is the relative square norm error between the two sinograms? Plot the two sinograms and their difference.