Building a matrix model for one-dimensional convolution

Samuli Siltanen

Department of Mathematics and Statistics
University of Helsinki, Finland
samuli.siltanen@helsinki.fi
www.siltanen-research.net

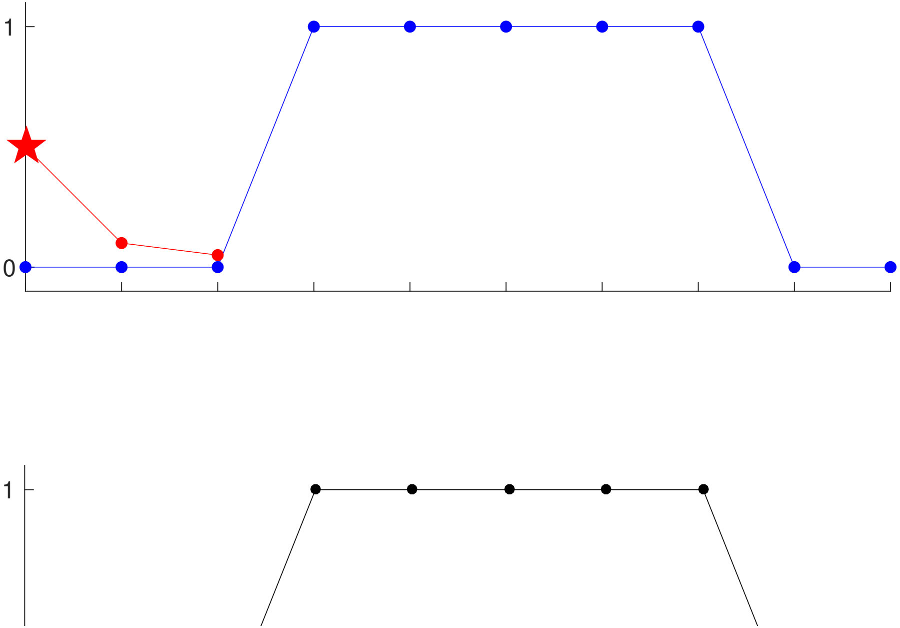
Inverse Problems Course
University of Helsinki, Finland
January 20, 2017

Outline

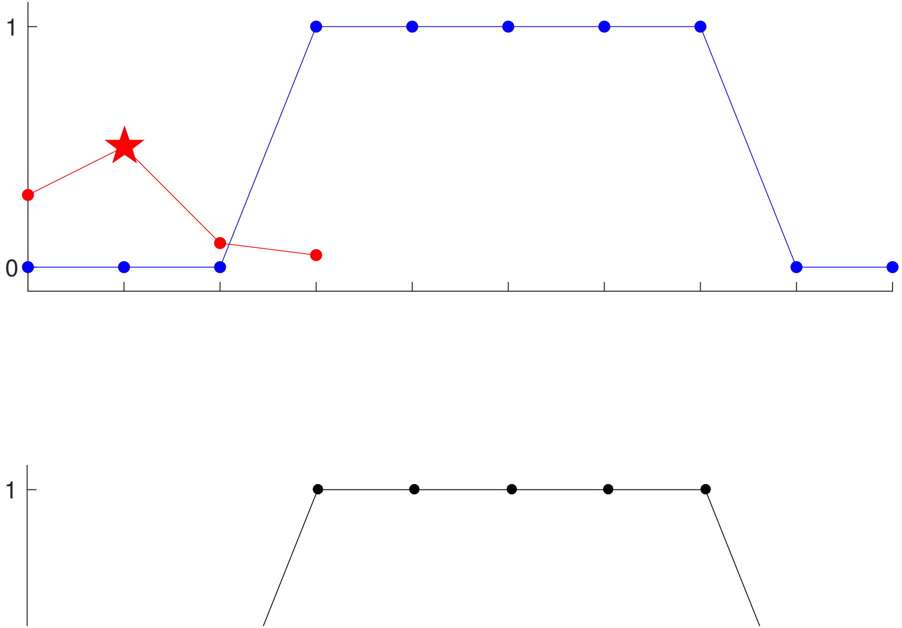
The convolution process

Constructing the matrix

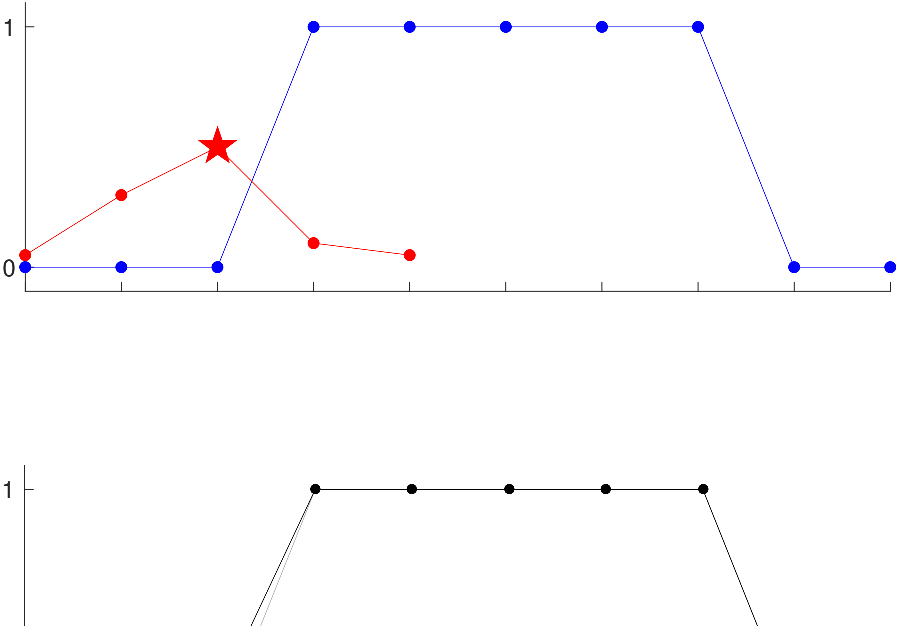
Convolution, position 1



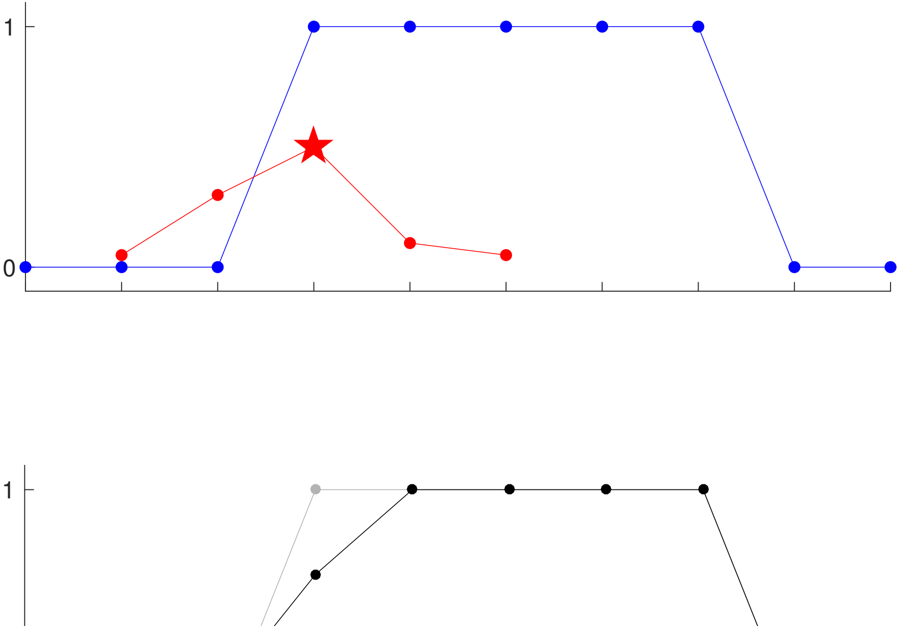
Convolution, position 2



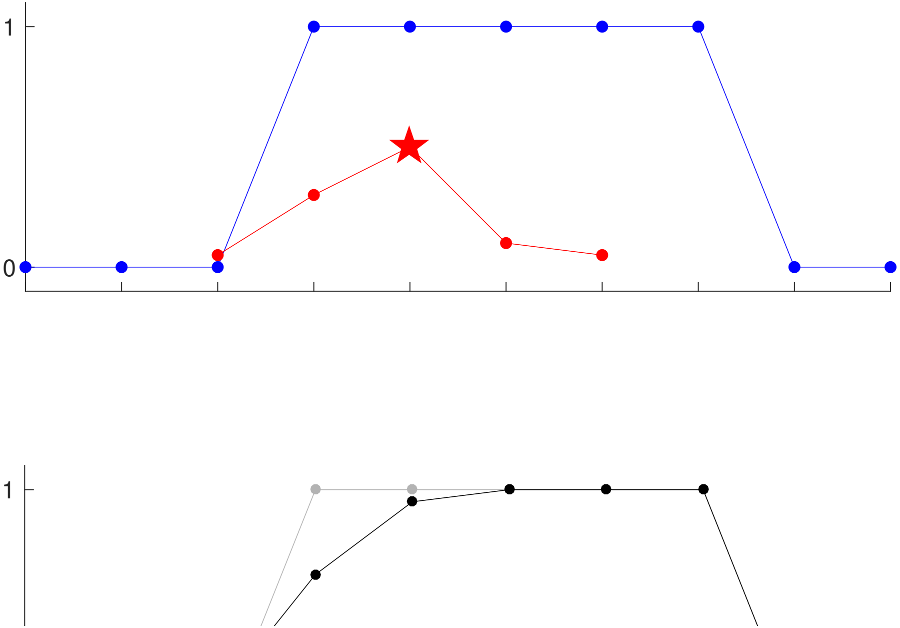
Convolution, position 3



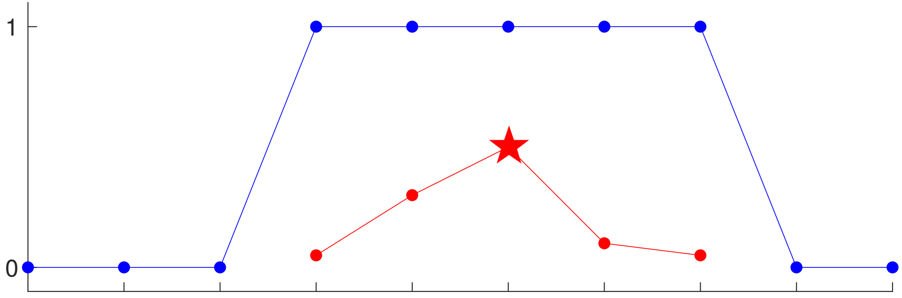
Convolution, position 4

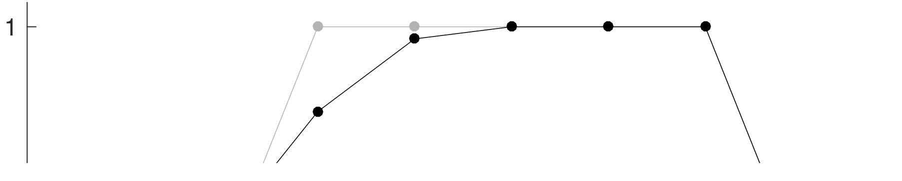


Convolution, position 5

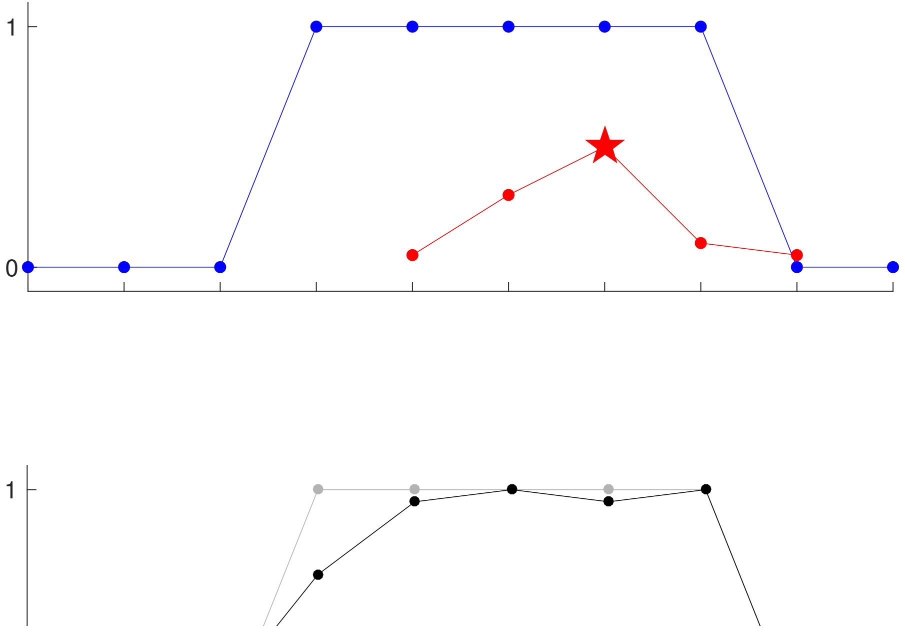


Convolution, position 6

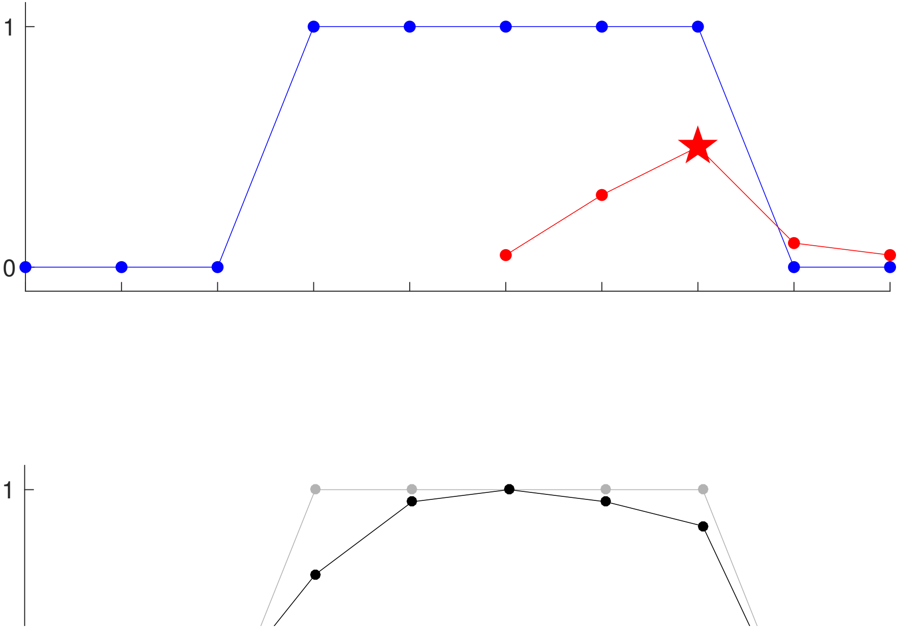




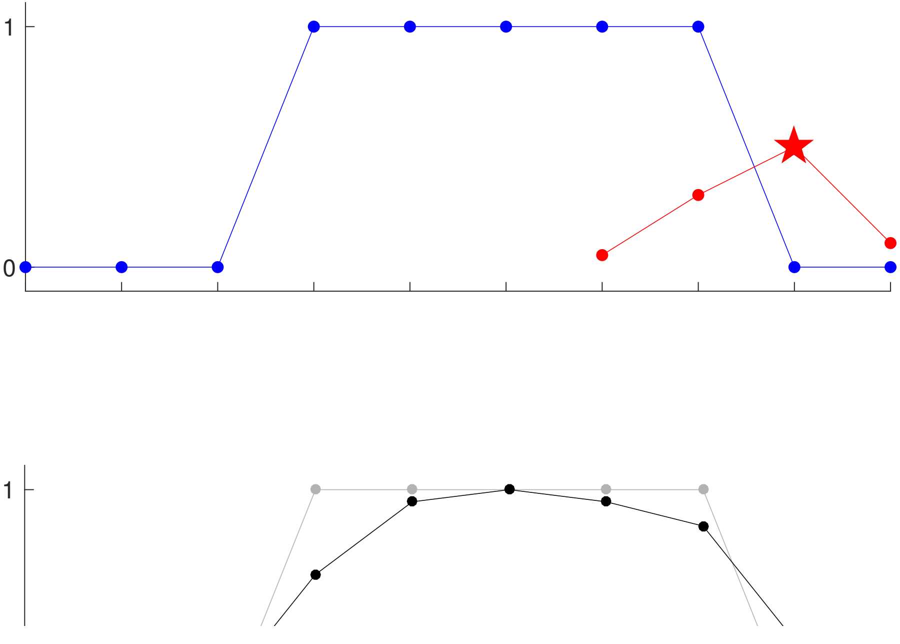
Convolution, position 7



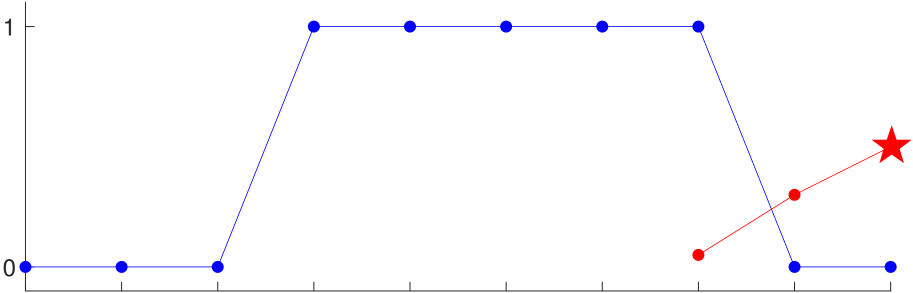
Convolution, position 8

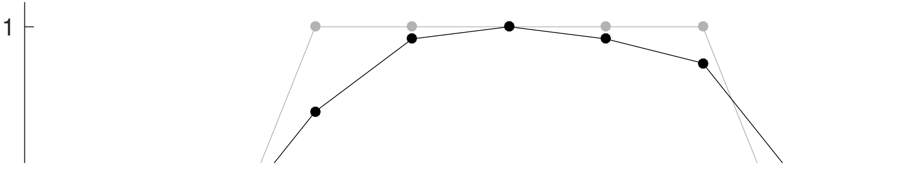


Convolution, position 9



Convolution, position 10





We define discrete convolution using periodic boundary conditions

Let $p \in \mathbb{R}^n$ and $s \in \mathbb{R}^n$. We call s the *signal* and p the *point spread function (PSF)*.

Convolution, denoted by $p * s \in \mathbb{R}^n$, is defined by the formula

$$(p * s)_j = \sum_{\ell=1}^{n} p_\ell s_{j-\ell}, \tag{1}$$

where $s_{j-\ell}$ is defined by periodic extension for the cases $j - \ell < 1$ and $j - \ell > n$. For example, $s_0 = s_n$ and $s_{-1} = s_{n-1}$ and $s_{n+1} = s_1$.

Step-by-step construction of the convolution vector (1) is shown in the following slides for a special case with $n = 10$. Note that only nonzero elements of $p \in \mathbb{R}^n$ are plotted. The elements $(p * s)_j$ are shown in blue color; the last slide (position 10) shows the complete vector $p * s \in \mathbb{R}^n$.

Outline

The convolution process

Constructing the matrix

Given a PSF vector p, the mapping $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\mathcal{A}(s) = p * s$ is linear

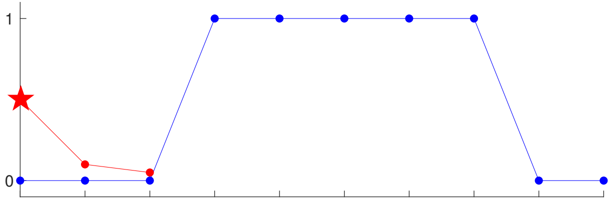
Therefore, we can construct a $n \times n$ matrix A so that

$$\mathcal{A}(s) = As.$$

The following slides demonstrate the construction of the matrix A. Note that the structure of A is highly redundant as the same PSF elements appear many times.

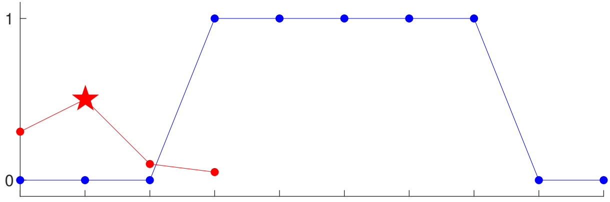
The Matlab command `convmtx.m` is useful for building convolution matrices.

Convolution matrix, positions up to 1



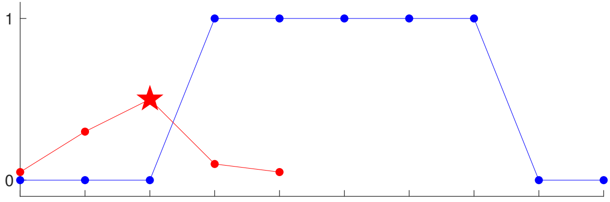
$$\begin{bmatrix} 0.50 & 0.10 & 0.05 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & & & \\ & & & & & & & & & \end{bmatrix}$$

Convolution matrix, positions up to 2



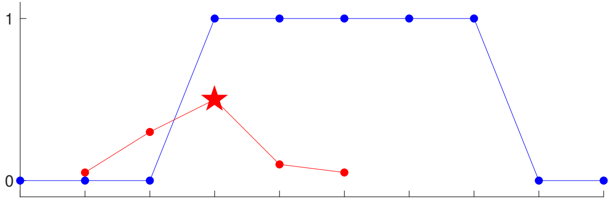
0.50	0.10	0.05	0	0	0	0	0	0	0
0.30	0.50	0.10	0.05	0	0	0	0	0	0

Convolution matrix, positions up to 3



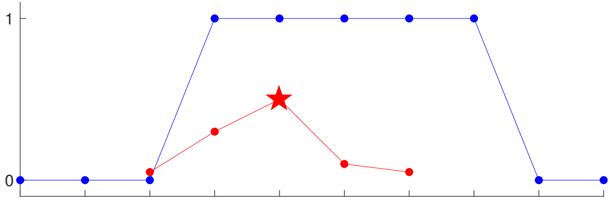
0.50	0.10	0.05	0	0	0	0	0	0	0
0.30	0.50	0.10	0.05	0	0	0	0	0	0
0.05	0.30	0.50	0.10	0.05	0	0	0	0	0

Convolution matrix, positions up to 4



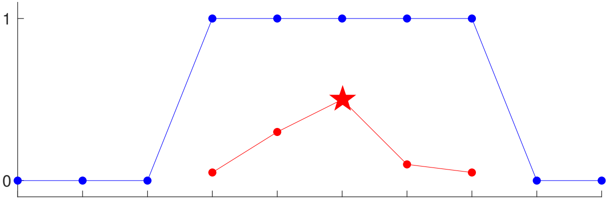
0.50	0.10	0.05	0	0	0	0	0	0	0
0.30	0.50	0.10	0.05	0	0	0	0	0	0
0.05	0.30	0.50	0.10	0.05	0	0	0	0	0
0	0.05	0.30	0.50	0.10	0.05	0	0	0	0

Convolution matrix, positions up to 5



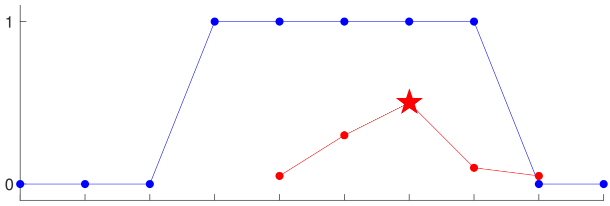
0.50	0.10	0.05	0	0	0	0	0	0	0
0.30	0.50	0.10	0.05	0	0	0	0	0	0
0.05	0.30	0.50	0.10	0.05	0	0	0	0	0
0	0.05	0.30	0.50	0.10	0.05	0	0	0	0
0	0	0.05	0.30	0.50	0.10	0.05	0	0	0

Convolution matrix, positions up to 6



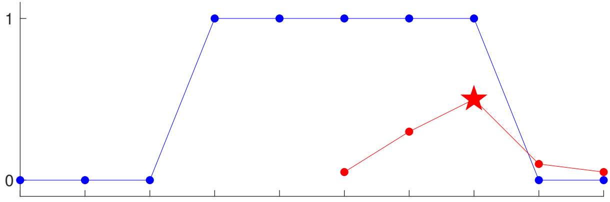
0.50	0.10	0.05	0	0	0	0	0	0	0
0.30	0.50	0.10	0.05	0	0	0	0	0	0
0.05	0.30	0.50	0.10	0.05	0	0	0	0	0
0	0.05	0.30	0.50	0.10	0.05	0	0	0	0
0	0	0.05	0.30	0.50	0.10	0.05	0	0	0
0	0	0	0.05	0.30	0.50	0.10	0.05	0	0

Convolution matrix, positions up to 7



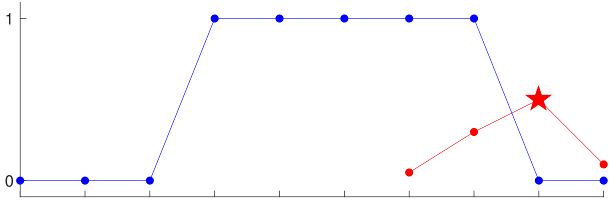
0.50	0.10	0.05	0	0	0	0	0	0	0
0.30	0.50	0.10	0.05	0	0	0	0	0	0
0.05	0.30	0.50	0.10	0.05	0	0	0	0	0
0	0.05	0.30	0.50	0.10	0.05	0	0	0	0
0	0	0.05	0.30	0.50	0.10	0.05	0	0	0
0	0	0	0.05	0.30	0.50	0.10	0.05	0	0
0	0	0	0	0.05	0.30	0.50	0.10	0.05	0

Convolution matrix, positions up to 8



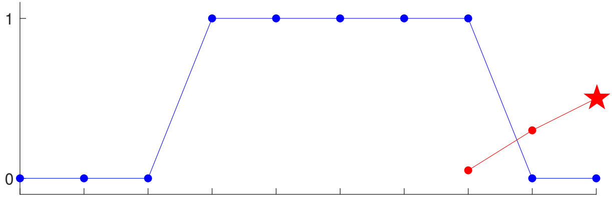
0.50	0.10	0.05	0	0	0	0	0	0	0
0.30	0.50	0.10	0.05	0	0	0	0	0	0
0.05	0.30	0.50	0.10	0.05	0	0	0	0	0
0	0.05	0.30	0.50	0.10	0.05	0	0	0	0
0	0	0.05	0.30	0.50	0.10	0.05	0	0	0
0	0	0	0.05	0.30	0.50	0.10	0.05	0	0
0	0	0	0	0.05	0.30	0.50	0.10	0.05	0
0	0	0	0	0	0.05	0.30	0.50	0.10	0.05

Convolution matrix, positions up to 9



0.50	0.10	0.05	0	0	0	0	0	0	0
0.30	0.50	0.10	0.05	0	0	0	0	0	0
0.05	0.30	0.50	0.10	0.05	0	0	0	0	0
0	0.05	0.30	0.50	0.10	0.05	0	0	0	0
0	0	0.05	0.30	0.50	0.10	0.05	0	0	0
0	0	0	0.05	0.30	0.50	0.10	0.05	0	0
0	0	0	0	0.05	0.30	0.50	0.10	0.05	0
0	0	0	0	0	0.05	0.30	0.50	0.10	0.05
0	0	0	0	0	0	0.05	0.30	0.50	0.10

Convolution matrix, positions up to 10



$$\begin{bmatrix} 0.50 & 0.10 & 0.05 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.30 & 0.50 & 0.10 & 0.05 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.05 & 0.30 & 0.50 & 0.10 & 0.05 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.05 & 0.30 & 0.50 & 0.10 & 0.05 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.05 & 0.30 & 0.50 & 0.10 & 0.05 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.05 & 0.30 & 0.50 & 0.10 & 0.05 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.05 & 0.30 & 0.50 & 0.10 & 0.05 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.05 & 0.30 & 0.50 & 0.10 & 0.05 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.05 & 0.30 & 0.50 & 0.10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.05 & 0.30 & 0.50 \end{bmatrix}$$