Introduction to mathematical physics: Quantum dynamics — Homework set 13, 5.5.2017

This is the final Homework set. During the lectures over the last week, on 2.5. and 4.5., there will be time for the project seminars, as well as for discussion and questions about any content of the course.

Exercise 1

The free Hamiltonian of a relativistic particle with a mass $m > 0$ (and using units in which the speed of light is one) is given by

$$H_{\mathrm{rel}} := \sqrt{-\nabla^2 + m^2}\,.$$

The operator is understood to be defined via the same construction as H_0 was in the lecture notes, i.e., using the corresponding multiplication operator in Fourier space. Hence, it defines a self-adjoint operator on the Hilbert space $L^2(\mathbb{R}^3)$.

The corresponding free Hamiltonian of a classical particle is given by $H_0 := -\frac{1}{2m}\nabla^2$. Show that $D(H_0) \subset D(H_{\mathrm{rel}})$, and that $\|(H_{\mathrm{rel}} - m)\psi\| \le \|H_0\psi\|$ for all $\psi \in D(H_0)$. (*Hint*: $(a+b)(a-b) = a^2 - b^2$ for all $a, b \in \mathbb{C}$.)

Exercise 2

Proof of Proposition 12.2.5.

Let $\mathfrak{h}$ be some Hilbert space, choose $N \in \mathbb{N}_+$, and consider the following subsets of the tensor product $\mathcal{H}_N := \bigotimes_{n=1}^{N} \mathfrak{h}$,

$$\mathcal{H}_N^{(+)} := \{\Psi \in \mathcal{H}_N \mid \Psi \text{ is totally symmetric}\}\,,$$
$$\mathcal{H}_N^{(-)} := \{\Psi \in \mathcal{H}_N \mid \Psi \text{ is totally antisymmetric}\}\,.$$

For both $\sigma = +1$ and $\sigma = -1$ prove that $\mathcal{H}_N^{(\sigma)}$ is a closed subspace of $\mathcal{H}_N$, and that the corresponding orthogonal projection $P_N^{(\sigma)}$ onto $\mathcal{H}_N^{(\sigma)}$ satisfies for all $\psi \in \mathfrak{h}^N$,

$$P_N^{(\sigma)}\Big(\bigotimes_{n=1}^{N} \psi_n\Big) = \frac{1}{N!}\sum_{\pi \in S_N} \sigma^\pi \bigotimes_{n=1}^{N} \psi_{\pi(n)}\,,$$

where S_N denotes the set of permutations of the index set $\{1, 2, \ldots, N\}$, $(-1)^\pi$ denotes the sign of the permutation $\pi \in S_N$, and $(+1)^\pi = 1$ for all permutations $\pi \in S_N$.

(*Hint:* Begin by recalling the basic definitions and properties listed in item 12.2.3 of the lecture notes about the properties of permutations, and the definition 12.2.4 of totally symmetric and antisymmetric vectors in $\bigotimes_{n=1}^{N} \mathfrak{h}$. Recall also that by Exercise 4.2 the orthogonal projection $P_N^{(\sigma)}$ is self-adjoint which will allow to conclude that $P_N^{(\sigma)}\Big(\bigotimes_{n=1}^{N} \psi_{\pi(n)}\Big) = \sigma^\pi P_N^{(\sigma)}\Big(\bigotimes_{n=1}^{N} \psi_n\Big)$.)

(Please turn over. . .)

Exercise 3

Let $\mathfrak{h}$ be a Hilbert space, and consider the standard Fock space generated by it: define $\mathcal{H}_0 = \mathbb{C}$, $\mathcal{H}_1 = \mathfrak{h}$, and $\mathcal{H}_N = \bigotimes_{n=1}^{N} \mathfrak{h}$, for $N = 2, 3, \ldots$, and then set $\mathcal{F} := \bigoplus_{n=0}^{\infty} \mathcal{H}_N$. Consider some fixed $g \in \mathfrak{h}$.

(a) For $N \in \mathbb{N}_+$ prove that there is a unique continuous linear map $a_N : \mathcal{H}_N \to \mathcal{H}_{N-1}$ with

$$a_N\Big(\bigotimes_{n=1}^{N} \psi_n\Big) = \sqrt{N}\, \langle g|\psi_1\rangle_{\mathfrak{h}} \bigotimes_{n=2}^{N} \psi_n\,, \qquad \text{for all } \psi \in \mathfrak{h}^N := \prod_{n=1}^{N} \mathfrak{h}\,.$$

(*Hint*: Theorem 2.12 and Exercise 2.4. Note that the above formula cannot be used directly as a definition for *all* $\psi \in \mathfrak{h}^N$ since some of them will map into the same vector $\bigotimes_{n=1}^{N} \psi_n$ in $\mathcal{H}_N$. However, recall that for any non-zero $f \in \mathfrak{h}$ one can find an orthonormal basis of $\mathfrak{h}$ which contains $f/\|f\|$.)

(b) Show that $D_0 := \left\{ \Psi \in \mathcal{F} \,\middle|\, \sum_{N=0}^{\infty} N\|\Psi_N\|^2 < \infty \right\}$ is a dense subspace of $\mathcal{F}$ which contains the *vacuum vector* $\Omega = (1, 0, 0, \ldots)$.

(c) Prove that the equation $(a\Psi)_N = a_{N+1}\Psi_{N+1}$, $N = 0, 1, \ldots$, defines an operator $D_0 \to \mathcal{F}$, and that this operator is *unbounded* if $g \neq 0$. Compute $a\Omega$.

(d) Show that there is a unique continuous linear map $c_N : \mathcal{H}_N \to \mathcal{H}_{N+1}$ with

$$c_N\Big(\bigotimes_{n=1}^{N} \psi_n\Big) = \sqrt{N+1}\, g \otimes \psi_1 \otimes \cdots \otimes \psi_N\,, \qquad \text{for all } \psi \in \mathfrak{h}^N\,,$$

for any choice of $N = 0, 1, \ldots$. Prove that if we set $(c\Psi)_0 = 0$ and $(c\Psi)_N = c_{N-1}\Psi_{N-1}$, for $N \in \mathbb{N}_+$, then c is an operator $D_0 \to \mathcal{F}$ which is unbounded if $g \neq 0$. Compute $c\Omega$.

$a = a(g)$ is called the *annihilation operator* related to g on $\mathcal{F}$ and $c = c(g)$ is called the *creation operator* related to g. Note that order is important here, and it would be better to say that the operators annihilate and create a particle with the label "1".

Exercise 4

Consider the fermionic Fock space defined in 12.2.6: $\mathcal{F}^{(-)} = \bigoplus_{N=0}^{\infty} \mathcal{H}_N^{(-)}$, where $\mathcal{H}_N^{(-)}$ is the totally antisymmetric subspace of $\mathcal{H}_N$. As in Exercise 2, let $P_N^{(-)}$ denote the orthogonal projection onto $\mathcal{H}_N^{(-)}$, and consider some fixed $g \in \mathfrak{h}$. The following statements show that the fermionic creation and annihilation operators, defined by restricting $a(g)$ and $c(g)$ to $\mathcal{F}^{(-)}$, are actually *bounded* operators.

(a) Show that the formulae $(P^{(-)}\Psi)_0 := \Psi_0$, $(P^{(-)}\Psi)_N := P_N^{(-)}\Psi_N$, for $N \in \mathbb{N}_+$, define an orthogonal projection $P^{(-)} : \mathcal{F} \to \mathcal{F}$ onto $\mathcal{F}^{(-)}$.

(b) Prove that $D_- := D_0 \cap \mathcal{F}^{(-)}$ is a dense subspace of $\mathcal{F}^{(-)}$, and consider the restrictions of $a(g)$ and $c(g)$ to $\mathcal{F}^{(-)}$, i.e., the maps $\tilde{a} := P^{(-)}a(g)|_{D_-}$ and $\tilde{c} := P^{(-)}c(g)|_{D_-}$. Show that there are unique $a_-(g), c_-(g) \in \mathcal{B}(\mathcal{F}^{(-)})$ such that $a_-(g)|_{D_-} = \tilde{a}$, $c_-(g)|_{D_-} = \tilde{c}$, and that then $\|a_-(g)\| = \|g\|_{\mathfrak{h}} = \|c_-(g)\|$. (*Hint*: What happens to $P_N^{(-)}(\otimes_{n=1}^{N}\psi_n)$, if $\psi_i = \psi_j$ for some $i \neq j$?)

(c) Show that $c_-(g)$ is the adjoint of $a_-(g)$. (In this context, usually denoted by $a_-^*(g)$.)