**Introduction to mathematical physics: Quantum dynamics**

Homework set 11 (21.4.2017)

**NB.** The date for the exercise session has not been fixed yet: please check the web page of the course for the time and place of the session.

## Exercise 1

### Proof of Proposition 10.3

Let $d \leq 3$, and consider $H_0 = -\frac{1}{2}\nabla^2$ on $L^2(\mathbb{R}^d)$. Prove that every $\psi \in D(H_0)$ can be chosen continuous. Show that for any $\varepsilon > 0$ there is $c_\varepsilon \geq 0$ such that for all $\psi \in D(H_0)$,

$$\sup_{x\in\mathbb{R}^d} |\psi(x)| = \|\psi\|_\infty \leq \varepsilon\|H_0\psi\| + c_\varepsilon\|\psi\| . \tag{1}$$

(*Hint*: Recall the Riemann-Lebesgue lemma; in particular, note that by Remark "a)" on p. 73 of the lecture notes to the inverse Fourier transform one clearly has $\|\psi\|_\infty \leq \|\mathcal{F}\psi\|_1$ for all $\psi \in L^2$. Show then that the assumptions imply $\mathcal{F}\psi \in L^1(\mathbb{R}^d)$, and try to prove the inequality (1) for some constants. After this, consider the functions $f_r(x) := \psi(r^{-1}x)$ with $r > 0$.)

## Exercise 2

Let $d = 1$, and consider $H_0 = -\frac{1}{2}\nabla^2$ on $L^2(\mathbb{R})$. By Exercise 1 every element of $D(H_0)$ is then a continuous function. However, they are even more regular:

(a) Show that, if $\psi \in D(H_0)$, then $\psi \in C^{(1)}(\mathbb{R})$ and with $\phi := H_0\psi$ we have for all $x, x_0 \in \mathbb{R}$

$$\psi(x) = \psi(x_0) + (x - x_0)\psi'(x_0) - 2\int_{x_0}^{x} \mathrm{d}y\, (x-y)\phi(y) . \tag{2}$$

(b) Conversely, show that, if $\psi \in C^{(1)}(\mathbb{R})$ is such that $\psi, \psi' \in L^2(\mathbb{R})$ and there are some $x_0 \in \mathbb{R}$ and $\phi \in L^2(\mathbb{R})$ such that (2) holds for almost every $x \in \mathbb{R}$, then $\psi \in D(H_0)$ and $\phi = H_0\psi$.

(*Hint*: Recall that $-2H_0 = \partial_x\partial_x$ and use Exercise 10.2.)

## Exercise 3

Let $d = 1$ and consider a real potential $V \in L^\infty(\mathbb{R})$ which is piecewise continuous: it is continuous apart from some isolated points in $Z \subset \mathbb{R}$. By Theorem 10.4, $H = H_0 + V$ is self-adjoint on $D(H_0)$.

(a) Suppose $\lambda \in \mathbb{R}$ is an eigenvalue of $H$ with an eigenvector $\psi$, that is, assume $\psi \in D(H)$ and $H\psi = \lambda\psi$. Show that then between any two consecutive points of discontinuity of $V$ (that is, on any open interval in $\mathbb{R} \setminus Z$) $\psi$ is twice continuously differentiable and satisfies the ordinary differential equation $-\frac{1}{2}\psi''(x) + V(x)\psi(x) = \lambda\psi(x)$. (*Hint*: Exercise 2.)

(b) State also the converse: Suppose $\lambda \in \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{C}$ satisfies $-\frac{1}{2}\psi''(x) + V(x)\psi(x) = \lambda\psi(x)$ on any open interval on which $V$ is continuous (in particular, $\psi''(x)$ exists at every point of such intervals). What do you still need to check so that $\psi$ is an eigenvector of $H$? Is it possible that the corresponding eigenvalue is not $\lambda$?

(Please turn over...)

## Exercise 4

Continuation from Exercise 3...

Consider $H$ in the special case $V(x) = -E_0 \mathbb{1}_{\{|x|<a\}}$ where $a, E_0 > 0$ are some given parameters. (This setup is called a one-dimensional *potential well*.)

Assume $\lambda$ is an eigenvalue of $H$, and find explicitly all of the corresponding eigenvectors. What is the dimension of the eigenspace? Derive also an implicit formula satisfied by any eigenvalue of $H$. Does $H$ have eigenvalues for all $a, E_0 > 0$?

## Exercise 5

**Free Schrödinger operator on $(0, 1)$**

In analogy to Exercise 10.4, define an operator $T$ on $\mathcal{H} := L^2(\Omega)$ with $\Omega := (0, 1)$ by using the domain

$$D(T) = \{\psi \in \mathcal{H} \,|\, \exists f \in D(H_0) \text{ such that } f|_\Omega = \psi\} ,$$

and setting $T\psi := H_0 f|_\Omega$ for $\psi \in D(T)$. Show that the definition makes sense: $T\psi$ does not depend on the choice of $f$.

For each of the domains $D$ listed below, consider the restriction $A := T|_D$ of $T$ to $D$. Show that every such $A$ is a symmetric densely defined operator.

(a) **(Dirichlet boundary conditions)**

$$D = \{\psi \in D(T) \,|\, \psi(0) = 0,\ \psi(1) = 0\} .$$

(b) **(Mixed Dirichlet and Neumann)** There is $\alpha \in \mathbb{R}$ such that

$$D = \{\psi \in D(T) \,|\, \psi'(0) = \alpha\psi(0),\ \psi(1) = 0\} , \quad \text{or}$$
$$D = \{\psi \in D(T) \,|\, \psi(0) = 0,\ \psi'(1) = \alpha\psi(1)\} .$$

(c) **(General Neumann)** There are $\alpha, \beta \in \mathbb{R}$ such that

$$D = \{\psi \in D(T) \,|\, \psi'(0) = \alpha\psi(0),\ \psi'(1) = \beta\psi(1)\} .$$

(d) **(Generalized periodic)** There are $\varphi \in \mathbb{R}$ and a $2 \times 2$ matrix $M$, such that $M_{ij} \in \mathbb{R}$, $i, j = 1, 2$ and $\det M = 1$, and we define

$$D = \left\{ \psi \in D(T) \middle| \begin{pmatrix} \psi(1) \\ \psi'(1) \end{pmatrix} = \mathrm{e}^{\mathrm{i}\varphi} M \begin{pmatrix} \psi(0) \\ \psi'(0) \end{pmatrix} \right\} .$$

(In fact, all of these operators are self-adjoint. Moreover, they yield the full collection of self-adjoint extensions in $L^2(\Omega)$ of the operator $f \mapsto -f''$ with the domain $C_c^\infty(\Omega)$. Therefore, they are all candidates for a possible definition of the generator of "free" evolution on $(0, 1)$. *Hint*: try first to figure out why $T$ is *not* self-adjoint.)