Introduction to mathematical physics: Quantum dynamics — Homework set 10, 7.4.2017

Exercise 1

"C-real" symmetric operators have self-adjoint extensions

A map $C : \mathcal{H} \to \mathcal{H}$ is called a *conjugation* if C is *conjugate linear*, it *swaps scalar products*, and $C^2 = 1$. (In other words, we assume $C(\alpha\psi) = \alpha^* C\psi$, $C(\psi + \phi) = C\psi + C\phi$, $\langle C\psi|C\phi\rangle = \langle\phi|\psi\rangle$, and $C(C\psi) = \psi$ for all $\alpha \in \mathbb{C}$, $\psi, \phi \in \mathcal{H}$.) For instance, $(C\psi)(x) = \psi(x)^*$ defines a conjugation on $L^2(\mathbb{R}^d)$.

Given a conjugation C, an operator S on $\mathcal{H}$ is called *C-real* if it commutes with C in the following precise sense: $CD(S) \subset D(S)$ and $SC\psi = CS\psi$ for all $\psi \in D(S)$.

Show that, if C is a conjugation and S is a symmetric operator which is C-real, then S has self-adjoint extensions.

(*Hint:* Consider what happens to orthonormal sets in the deficiency spaces.)

(Please turn over...)

Exercise 2

(a) Suppose $\psi \in L^2(\mathbb{R})$ is "L^2-differentiable", that is, $\psi \in D(\partial)$. Let $\phi = \partial\psi$. Show that ψ can be chosen as a continuous function on $\mathbb{R}$, and that then for *all* $x', x \in \mathbb{R}$,

$$\psi(x') - \psi(x) = \int_x^{x'} \mathrm{d}y\, \phi(y)\,. \tag{1}$$

(The above integral notation means a directed integral from x to x': if $x \le x'$, we have $\int_x^{x'} \mathrm{d}y = \int_{[x,x']} \mathrm{d}y$, and if $x' < x$, then $\int_x^{x'} \mathrm{d}y = -\int_{[x',x]} \mathrm{d}y$.)
(*Hint:* Using Hölder's inequality, show first that $\mathcal{F}\psi \in L^1(\mathbb{R})$. This implies that ψ is continuous. Recall that by Proposition 6.6.4 we have now $(f', \psi) = -(f, \phi)$ for any Schwartz function f. Consider then the integral $\int_{\mathbb{R}} \mathrm{d}x\, f(x)^*(\psi(x+a) - \psi(x))$ for an arbitrary compactly supported test function f and any $a > 0$ and use continuity in x.)

(b) Prove also the converse: if $x \in \mathbb{R}$ and $\phi, \psi \in L^2(\mathbb{R})$ are such that (1) holds for almost every $x' \in \mathbb{R}$, then $\psi \in D(\partial)$ and $\phi = \partial\psi$.
(*Hint:* Study $(f', \psi) + (f, \phi)$ for an arbitrary Schwartz function f.)

You will need these results later; if you get stuck, assume them to be proven and continue with the following exercises.

Exercise 3

Consider the open interval $\Omega := (0, 1) \subset \mathbb{R}$. Set $D_c := C_c^\infty(\Omega)$ and $\mathcal{H} := L^2(\Omega)$, and define $(S\psi)(x) := -\mathrm{i}\psi'(x)$ for $\psi \in D_c$.

(a) Show that S is a densely defined symmetric operator on $\mathcal{H}$.

(b) Hence, S is closable and let $\bar{S}$ denote its closure. For any $\psi \in \mathcal{H}$ let $f_\psi \in L^2(\mathbb{R})$ denote its null extension to the whole of $\mathbb{R}$: set $f_\psi(x) = \psi(x)$ for $x \in \Omega$ and $f_\psi(x) = 0$ otherwise. Show that $D(\bar{S}) = \{\psi \in \mathcal{H} \,|\, f_\psi \in D(\partial)\}$ and $\bar{S}\psi = -\mathrm{i}\partial f_\psi|_\Omega$ for all $\psi \in D(\bar{S})$.

(*Hint:* Exercise 2. It is probably better not to use Cayley transforms here. Note that if the function ϕ in (1) has a compact support, then the integral defines a function ψ which is *constant* when restricted to the left of the support, as well as to the right of the support.)

Exercise 4

Continuation from Exercise 3...

Show that S is not essentially self-adjoint but it has self-adjoint extensions. Derive the following parameterization of the extensions: to each $\varphi \in [0, 2\pi)$ there corresponds a self-adjoint operator A_φ such that $S \subset A_\varphi$,

$$D(A_\varphi) = \left\{\psi \in \mathcal{H} \,\middle|\, \exists f \in D(\partial) \text{ such that } f|_\Omega = \psi \text{ and } f(1) = \mathrm{e}^{\mathrm{i}\varphi} f(0)\right\}\,, \tag{2}$$

and for any such f we have $A_\varphi\psi = -\mathrm{i}\partial f|_\Omega$. Can you guess what is the unitary semigroup generated by A_φ on $\mathcal{H}$? (No need to try to prove your guess.)
(Note that by Exercise 2, if $f \in D(\partial)$, it makes sense to talk about "$f(x)$" for a fixed x.)

(*Hint:* Cayley transform. Note that if $f \in D(\partial)$ is such that $f|_\Omega$ belongs to one of the deficiency spaces, using Proposition 6.6.4 it follows that $\partial f|_\Omega$ is continuous. Then, by (1), f is continuously differentiable on Ω. This will imply that f satisfies an ordinary differential equation on Ω. Solve the equation.)