Introduction to mathematical physics: Quantum dynamics Homework set 9 31.3.2017

NB. Please note that there will be **no lectures nor "Ratkomo" tutorials on Thursday 30.3**. The exercise session on 31.3. will be held as usual.

Exercise 1

(a) The time-evolution x_t of a free *classical* particle is given by the unique twice continuously differentiable solution to the equation $\frac{\mathrm{d}^2}{\mathrm{d}t^2}x_t = 0$. Solve this equation assuming that the particle is initially, when $t = 0$, at $x_0 \in \mathbb{R}^d$ with a velocity $v_0 \in \mathbb{R}^d$, and denote the corresponding position at time t by $x_t(x_0, v_0)$ and the corresponding velocity by $v_t(x_0, v_0)$. (Recall that the velocity is defined by $v_t := \frac{\mathrm{d}}{\mathrm{d}t}x_t$.)

(b) Let $\mu_0 = \mu_0(\mathrm{d}x, \mathrm{d}v)$ be a probability measure on $\mathbb{R}^d \times \mathbb{R}^d$, and assume that the initial data (x_0, v_0) is distributed according to μ_0. Choose an arbitrary test-function observable $f \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ and denote $\langle f\rangle_t := \int \mu_0(\mathrm{d}x_0, \mathrm{d}v_0) f(x_t(x_0, v_0), v_t(x_0, v_0))$. Then $\partial_t\langle f\rangle_t = \langle g\rangle_t$ for some $g \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$. Find a formula for g.

(c) Consider then the special case in which $\mu_0(\mathrm{d}x, \mathrm{d}v) = \mathrm{d}x\mathrm{d}v\, P_0(x, v)$ for some $P_0 \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$. Show that then for all t there is $P_t \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ such that $\langle f\rangle_t = \int \mathrm{d}x\mathrm{d}v\, P_t(x, v) f(x, v)$ for all $f \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$. Here P_t satisfies a differential equation, what is it? (P_t is called the *classical phase space density* at time t.)

Exercise 2

Proof of Theorem 8.3.7.

Let $\psi(t)$, $t \in \mathbb{R}$, denote the solution to the free Schrödinger equation in $\mathbb{R}^d$ with initial data given by $\psi_0 \in L^2(\mathbb{R}^d)$, i.e., let $\psi(t) = \mathrm{e}^{-\mathrm{i}tH_0}\psi_0$. For $t \in \mathbb{R}$, let Λ_t denote the Wigner transform $\mathcal{W}_{\psi(t)}$ of $\psi(t)$.

Show that then

$$\partial_t\Lambda_t(x, k) + 2\pi k \cdot \nabla_x\Lambda_t(x, k) = 0\,. \tag{1}$$

Explicitly, you need to show that for all $t \in \mathbb{R}$ and $f \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$,

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{d}x\mathrm{d}k\, W[\psi(t)](x, k) f(x, k) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{d}x\mathrm{d}k\, W[\psi(t)](x, k)\; 2\pi k \cdot \nabla_x f(x, k)\,, \tag{2}$$

where $W[\psi(t)](x, k)$ is the Wigner function of $\psi(t)$. Can you also solve the equation, that is, write $W[\psi(t)](x, k)$ in terms of $W[\psi_0](x, k)$? Compare the result to Exercise 1.

(Please turn over...)

Exercise 3

Introduction to Cayley transforms

Assume $V : \mathbb{R}^d \to \mathbb{R}$ is Lebesgue measurable. By Exercise 5.1, the multiplication operator M_V is then self-adjoint on $L^2(\mathbb{R}^d)$. Let $\mathcal{C}$ denote the Cayley transform of M_V. Show that $\mathcal{C}$ is a multiplication operator, that is, find a Lebesgue measurable function $F : \mathbb{R}^d \to \mathbb{C}$ such that $\mathcal{C} = M_F$. Check by explicit computation that $|F(x)| = 1$ for (almost) all $x \in \mathbb{R}^d$ and use this to conclude that $\mathcal{C}$ is unitary.

Exercise 4

Let ε_n be a sequence for which $\varepsilon_n \to 0$ when $n \to \infty$, assume a sequence (ψ_n) in $L^2(\mathbb{R}^d)$ has been given, and consider the corresponding rescaled Wigner transforms $\Lambda_n = \mathcal{W}^{\varepsilon_n}_{\psi_n} \in \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$, as given in Definition 8.3.1.

Let $\phi \in \mathcal{S}(\mathbb{R}^d)$ be given, and consider the following explicit sequences of wave-vectors:

(a) $\psi_n = \phi$ for all n. Show that then $\Lambda_n \to \Lambda$ with $\Lambda(x,k) = \delta(x)|\widehat{\phi}(k)|^2$.

(b) $\psi_n = \phi^{\varepsilon_n}$ where $\phi^{\varepsilon}(x) := \varepsilon^{d/2}\phi(\varepsilon x)$.
Show that then $\Lambda_n \to \Lambda$ with $\Lambda(x,k) = |\phi(x)|^2\delta(k)$.

(*Hint:* Dominated convergence naturally, but applied *carefully.* The convergence of distributions above is defined by convergence using a fixed testfunction, i.e., $\Lambda_n \to \Lambda \Leftrightarrow \Lambda_n[f] \to \Lambda[f]$ for all $f \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$.)

Exercise 5

In this exercise we consider the Weyl quantization of some basic symbols $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$. Let $\nu \in \{1, 2, \dots, d\}$ denote a coordinate index.

(a) Show that Weyl quantization of the symbol $a(\mathbf{x}, \mathbf{k}) := x_\nu$ is given by the multiplication operator M_V for $V(\mathbf{x}) := x_\nu$.

(b) Show that Weyl quantization of the symbol $a(\mathbf{x}, \mathbf{k}) := 2\pi k_\nu$ is given by the differential operator $\widehat{p}_\nu := -\mathrm{i}\partial_\nu$.