Introduction to mathematical physics: Homework set 7
Quantum dynamics **17.3.2017**

Reminder: This week is a semester break and there will be no lectures or exercise sessions. These exercises are due to Friday after the break.

NB: I have listed on the course web page some possible *topics for the final project* which will replace the exam for this course. As we discussed during the last lecture on Thursday, in addition to the project, a minimal amount of points from the exercises *or* an oral exam later in May will be required to get the credits from the course.

Exercise 1

Semigroups and generators under unitary transforms

Suppose (U_t), $t \geq 0$, is a strongly continuous unitary semi-group with an infinitesimal generator A. Let V be an arbitrary unitary operator and define $\tilde{U}_t = V^* U_t V$ and $\tilde{A} = V^* A V$ (the operator $\tilde{A}$ is defined using the natural domain of a product of unbounded operators; see the lectures notes, Definition 5.17.)

Prove that $D(\tilde{A}) = V^* D(A)$. Show that $(\tilde{U}_t)$ is a strongly continuous unitary semi-group whose infinitesimal generator is $\tilde{A}$.

Exercise 2

Unitary extensions of isometries

Let T be an operator with domain $D = D(T)$ and range $R = R(T)$. Assume T is an *isometry*: $\|T\psi\| = \|\psi\|$ for all $\psi \in D$.

(a) Show that T has a unique continuous extension $\overline{T} : \overline{D} \to \overline{R}$, which is also an isometry. Let P denote the orthogonal projection onto $\overline{D}$, and define $V\psi = \overline{T}(P\psi)$ for all $\psi \in \mathcal{H}$. Then $V \in \mathcal{B}(\mathcal{H})$, and V is called a *partial isometry* on $\mathcal{H}$.

(b) Assume that $U : \mathcal{H} \to \mathcal{H}$ is a unitary extension of T, and denote $V' = U - V$. Show that $V'\psi = 0$ for all $\psi \in \overline{D}$. Denote the restriction of V' to $D^{\perp}$ by W. Show that W is a Hilbert space isomorphism between $D^{\perp}$ and $R^{\perp}$.

(c) Does every isometry have unitary extensions?

(Hint: Exercise 2.4.)

Remark: The notations $V = \overline{T} \oplus 0$ and $U = \overline{T} \oplus W$ can be used to denote the constructions in (a) and (b), respectively.

Exercise 3

Consider $\mathcal{H} := L^2(\mathbb{R}^d)$. Suppose $v_0 \in \mathbb{R}^d$ is given and define $(U_t\psi)(x) := \psi(x - tv_0)$ for $t \geq 0$, $x \in \mathbb{R}^d$, and $\psi \in \mathcal{H}$. Prove that the family U_t forms a strongly continuous unitary semigroup on $\mathcal{H}$. Consider the induced evolution and suppose you know that the particle is initially localized near a point $x_0 \in \mathbb{R}^d$. What can you say about its position at time t? What is the infinitesimal generator of the semigroup? (Hint: You can first try to find the generator on a dense set and then show that it is essentially self-adjoint there.)

(Please turn over)

Exercise 4a

Multi-indices

Suppose f, g are smooth functions on some open subset of $\mathbb{R}^d$. Show that for any multi-index $\alpha \in \mathbb{N}_0^d$ the generalized *Leibniz rule* holds:

$$\partial^\alpha (fg) = \sum_{\beta:\beta\leq\alpha} \binom{\alpha}{\beta} (\partial^{\alpha-\beta} f)(\partial^\beta g)\,.$$

Explanation of the notations: the sum goes over multi-indices β for which $\beta_i \leq \alpha_i$ for all $i = 1, 2, \ldots, d$, and $\binom{\alpha}{\beta} := \frac{\alpha!}{\beta!(\alpha-\beta)!}$, where $\alpha! := \prod_{i=1}^d (\alpha_i!)$.

Exercise 4b

Fourier transform maps derivatives to multiplications by polynomials

Let $\mathcal{S} = \mathcal{S}_d$ be the space of Schwartz test-functions on $\mathbb{R}^d$, with $d \geq 1$ (that is, $\mathcal{S}$ is the space of rapidly decreasing functions on $\mathbb{R}^d$), and let $\mathcal{F}$ denote the Fourier transform on $\mathcal{S}$ as defined in the lecture notes. Show that then for any multi-index α and $f \in \mathcal{S}$,

$$\mathcal{F}(\partial^\alpha f)(k) = (\mathrm{i}2\pi k)^\alpha (\mathcal{F}f)(k), \qquad \text{for all } k \in \mathbb{R}^d\,.$$

Exercise 5

Let $\mathcal{S}$ and $\mathcal{F}$ be given as in Exercise 4b. Define for all $t \in \mathbb{R}$ the map V_t by

$$(V_t f)(k) := \mathrm{e}^{-\mathrm{i}t\frac{1}{2}(2\pi k)^2} f(k), \qquad k \in \mathbb{R}^d\,.$$

Prove that V_t maps $\mathcal{S}$ to itself. Therefore, we can define the maps $U_t : \mathcal{S} \to \mathcal{S}$, $t \in \mathbb{R}$, by $U_t := \mathcal{F}^{-1} V_t \mathcal{F}$. Show that U_t has the following integral representation for $t \neq 0$: if $f \in \mathcal{S}$,

$$(U_t f)(x) = \int_{\mathbb{R}^d} \mathrm{d}y\, K_t(x-y) f(y), \qquad \text{for all } x \in \mathbb{R}^d\,,$$

where

$$K_t(x) = \left(\frac{1}{\sqrt{\mathrm{i}2\pi t}}\right)^d \mathrm{e}^{\mathrm{i}\frac{1}{2t}x^2}\,.$$

Here $\sqrt{\cdot}$ denotes the principal branch of the square root: for any $z \in \mathbb{C}$, $z \neq 0$, there are unique $r > 0$ and $\varphi \in (-\pi, \pi]$ such that $z = r\mathrm{e}^{\mathrm{i}\varphi}$; we define then $\sqrt{z} = r^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\varphi/2}$ and $\sqrt{0} = 0$.

(Warning! Do not change order of integration without checking that you can apply Fubini's theorem. Hint: Try to regularize the integral so that you can apply the following one-dimensional Gaussian integral: if $w \in \mathbb{C}$ and $z \in \mathbb{C}$ is such that $\mathrm{Re}\, z > 0$, then

$$\int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{\mathrm{i}2\pi k w - z\frac{1}{2}(2\pi k)^2} = \frac{1}{\sqrt{2\pi z}} \mathrm{e}^{-\frac{1}{2z}w^2}\,.$$

You can assume this integral to be known, or try to prove it yourself.)