Introduction to mathematical physics: Quantum dynamics — Homework set 6, 3.3.2017

In all of the exercises below, if T is a normal operator and $f : \sigma(T) \to \mathbb{C}$ is a Borel function, the notation "$f(T)$" means the operator defined using the spectral decomposition as explained in Theorem 5.24 in the lecture notes.

Exercise 1

Consider a projection valued measure E on the set X acting on the Hilbert space $\mathcal{H}$. Show that if $h : X \to \mathbb{C}$ is measurable and $|h(x)| = 1$ for all $x \in X$, then the operator $\mathcal{O}(h)$ is unitary. (This result was used in the Stone's theorem.)

Using this show that, if $\phi, \psi \in \mathcal{H}$, then the total variation $|E_{\phi,\psi}|$ of the complex measure $E_{\phi,\psi}$ satisfies $|E_{\phi,\psi}|(X) \le \|\phi\|\|\psi\|$. Conclude that therefore $\|\mathcal{O}(h)\| \le \sup_x |h(x)|$ for any measurable function h.

(The aim of the exercise is to get familiar with basic properties of PVMs. It is not supposed to be technically difficult: you are very welcome to use basic results from measure theory in the proofs, such as the *polar decomposition* for a complex measure.)

Exercise 2

Consider a normal operator T on a Hilbert space $\mathcal{H}$. Suppose f_n, $n \in \mathbb{N}$, is a sequence of bounded Borel functions such that $\sup_{n,z} |f_n(z)| < \infty$ and $f_n(z) \to f(z)$ for all $z \in \mathbb{C}$. Show that then $f_n(T) \to f(T)$ in the strong operator topology. Is this always true if we only assume $f_n(z) \to f(z)$ Lebesgue almost everywhere?

Exercise 3

Show that T is a normal operator if and only if it is a densely defined operator which satisfies $D(T^*) = D(T)$ and $\|T^*\psi\| = \|T\psi\|$ for all $\psi \in D(T)$.
(*Hint*: Unlike in Rudin's book, you are allowed to use here the fact that every normal operator (as defined in the lecture notes and in Rudin) has a spectral decomposition, as explained in Theorem 5.24. This will help in proving one of the directions.)

(Please turn over)

Exercise 4

Suppose A is a *bounded* self-adjoint operator and let E denote its spectral decomposition. Suppose $a, b \in \mathbb{R}$, $a < b$, and that $\varepsilon_n > 0$, $n \in \mathbb{N}$, form a sequence such that $\varepsilon_n \to 0$ as $n \to \infty$. Define for $\lambda \in \mathbb{R}$, $n \in \mathbb{N}$,

$$f_n(\lambda) := \frac{1}{\pi} \int_a^b \mathrm{d}s \, \mathrm{Im} \left(\frac{1}{\lambda - s - \mathrm{i}\varepsilon_n} \right) ,$$

where $\mathrm{Im}\, z$ denotes the imaginary part of the complex number z. Consider the two Borel sets (intervals) $I := (a, b)$ and $\overline{I} := [a, b]$. Show that $f_n(A) \to \frac{1}{2}\left(E(I) + E(\overline{I})\right)$ as $n \to \infty$ in the strong operator topology.

(This tells how to recover the spectral projections of self-adjoint operators if the resolvent map is known. *Hint*: Exercise 2.)

Exercise 5

Let $\Omega \subset \mathbb{R}^d$ be a non-empty Lebesgue measurable set and define $\mathcal{H} := L^2(\Omega)$. Assume that $K : \Omega \times \Omega \to \mathbb{C}$ is Lebesgue measurable. Define for any $\psi \in \mathcal{H}$ first a subset

$$X_\psi := \left\{ x \in \Omega \,\middle|\, \int_\Omega \mathrm{d}y \, |K(x, y)\psi(y)| < \infty \right\} , \tag{1}$$

and then a function $F_\psi : \Omega \to \mathbb{C}$ by

$$F_\psi(x) := \begin{cases} \int_\Omega \mathrm{d}y \, K(x, y)\psi(y), & \text{if } x \in X_\psi, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

An application of Fubini's theorem shows that F_ψ is Lebesgue measurable. Set

$$D(I_K) := \left\{ \psi \in \mathcal{H} \,\middle|\, \Omega \setminus X_\psi \text{ has zero measure and } \int_\Omega \mathrm{d}x \, |F_\psi(x)|^2 < \infty \right\} , \tag{3}$$

and let I_K denote the mapping $D(I_K) \to \mathcal{H}$ defined by $I_K\psi = F_\psi$. (More precisely, $I_K\psi$ is the equivalence class of F_ψ in $L^2(\Omega)$.)

(a) Show that I_K is an operator. (I_K is called the *integral operator* corresponding to the *integral kernel* K.)

Assume then $C_1 := \sup_{x\in\Omega} \int_\Omega \mathrm{d}y |K(x, y)| < \infty$ and $C_2 := \sup_{y\in\Omega} \int_\Omega \mathrm{d}x |K(x, y)| < \infty$. In this special case:

(b) Show that $I_K \in \mathcal{B}(\mathcal{H})$ and $\|I_K\| \le \sqrt{C_1 C_2}$.
(*Hint*: Square roots and Hölder's inequality.)

(c) Find the adjoint $(I_K)^*$.