Introduction to mathematical physics: Quantum dynamics — Homework set 4, 17.2.2017

Exercise 1

Consider $\mathcal{H} = \mathbb{C}^N$ and some matrix $M \in \mathbb{C}^{N\times N}$. Using the matrix exponentiation (as defined in Exercise 1.1) set $U_t := \mathrm{e}^{-\mathrm{i}tM}$ for $t \geq 0$. Show that the family (U_t) then forms a strongly continuous semigroup on $\mathcal{H}$: it satisfies "a)", "c)" and "d)" in Definition 4.1 of the lecture notes. Characterise all M for which the family (U_t) is a strongly continuous *unitary* semigroup.

Exercise 2

Assume that $P \in \mathcal{B}(\mathcal{H})$ is a non-zero projection. Prove that then all of the following statements are equivalent:

(a) P is an orthogonal projection ($R(P) = \mathrm{Ker}(P)^\perp$).
(b) P is self-adjoint.
(c) P is positive (see Exercise 3.1.c for the definition).
(d) P is normal.
(e) $\|P\| = 1$.
(f) $\|P\| \leq 1$.

Exercise 3

In addition to their norm-topologies, the spaces $\mathcal{H}$ and $\mathcal{B}(\mathcal{H})$ have other topologies which often appear in applications. Try to prove as many as you can of the following characterizations for the convergence of sequences under the relevant topologies introduced in the lecture notes:

Suppose $(\psi_n)_{n\in\mathbb{N}}$ is a sequence of vectors in $\mathcal{H}$. Then it converges to $\psi \in \mathcal{H}$

in norm, if $\|\psi - \psi_n\| \to 0$ when $n \to \infty$.
weakly, if $\langle\phi|\psi - \psi_n\rangle \overset{n\to\infty}{\longrightarrow} 0$ for all $\phi \in \mathcal{H}$.

A sequence of bounded operators, $T_n \in \mathcal{B}(\mathcal{H})$, $n \in \mathbb{N}$, converges to $T \in \mathcal{B}(\mathcal{H})$

uniformly, or *in norm*, if $\|T - T_n\|_{\mathcal{B}(\mathcal{H})} \to 0$ when $n \to \infty$.
strongly, or *in the strong operator topology*, if $\|T\psi - T_n\psi\|_{\mathcal{H}} \overset{n\to\infty}{\longrightarrow} 0$ for all $\psi \in \mathcal{H}$.
weakly, or *in the weak operator topology*, if $\langle\phi|T\psi - T_n\psi\rangle \overset{n\to\infty}{\longrightarrow} 0$ for all $\phi, \psi \in \mathcal{H}$.

We will use the following notations for the various types of convergence: "lim" or "$\to$" for norm-convergence, "s-lim" or "$\overset{\mathrm{s}}{\to}$" for strong convergence, and "w-lim" or "$\overset{\mathrm{w}}{\to}$" for weak convergence. Show that

(a) $\psi_n \to \psi \quad \Longrightarrow \quad \psi_n \overset{\mathrm{w}}{\to} \psi$.
(b) $T_n \to T \quad \Longrightarrow \quad T_n \overset{\mathrm{s}}{\to} T \quad \Longrightarrow \quad T_n \overset{\mathrm{w}}{\to} T$.

(Please turn over)

Exercise 4

Let $\mathcal{H} = \ell_2(\mathbb{N})$, that is, let $\mathcal{H} = L^2(\mu)$ with μ the counting measure on $\mathbb{N} = \{0, 1, \ldots\}$. Consider the following mappings defined for $\psi \in \mathcal{H}$, $n \geq 1$, and $k \in \mathbb{N}$:

$$(T_n^{(1)}\psi)(k) = \frac{1}{n}\psi(k),$$

$$(T_n^{(2)}\psi)(k) = \begin{cases} \psi(k), & \text{if } k \geq n, \\ 0, & \text{otherwise,} \end{cases}$$

$$(T_n^{(3)}\psi)(k) = \begin{cases} \psi(k-n), & \text{if } k \geq n, \\ 0, & \text{otherwise.} \end{cases}$$

(In other words, $T_n^{(1)}$ is division by n, $T_n^{(2)}$ is cancellation of the first n elements, and $T_n^{(3)}$ is right-shift by n.)

Prove that

(a) $T_n^{(i)} \in \mathcal{B}(\mathcal{H})$ for $i = 1, 2, 3$ and $n \geq 1$.
(b) $T_n^{(1)} \to 0$ in norm.
(c) $T_n^{(2)} \overset{\mathrm{s}}{\to} 0$ but $T_n^{(2)} \not\to 0$.
(d) $T_n^{(3)} \overset{\mathrm{w}}{\to} 0$ but $T_n^{(3)} \overset{\mathrm{s}}{\not\to} 0$ and $T_n^{(3)} \not\to 0$.

Exercise 5

States and C^*-algebras: an example

Let ν be a Borel probability measure on the unit sphere S of a complex Hilbert space $\mathcal{H}$ (i.e., $S := \{\psi \in \mathcal{H} \,|\, \|\psi\| = 1\}$).

(a) Show that there is a unique operator $\rho \in \mathcal{B}(\mathcal{H})$ which satisfies

$$\langle \phi_1 | \rho \phi_0 \rangle = \int_S \nu(\mathrm{d}\psi) \, \langle \phi_1 | \psi \rangle \, \langle \psi | \phi_0 \rangle$$

for all $\phi_0, \phi_1 \in \mathcal{H}$. Conclude that necessarily $\rho \geq 0$ (i.e., ρ is a positive operator). (Hint: Section 3 of the lecture notes.)

(b) Show that if $\mathcal{H} = \mathbb{C}^d$, $d < \infty$, then $\operatorname{Tr} \rho = 1$, and for all $A \in \mathcal{B}(\mathcal{H})$,

$$\operatorname{Tr}(\rho A) = \int_S \nu(\mathrm{d}\psi) \, \langle \psi | A \psi \rangle \, .$$

For any finite collection F of pairwise disjoint unit vectors $\psi_i \in \mathcal{H}$, $i = 1, 2, \ldots, N$, (i.e., we assume $\|\psi_i\| = 1$ for all i and $\psi_i \neq \psi_j$ for $i \neq j$) we can define a Borel probability measure ν_F on S by setting

$$\nu_F(f) = \frac{1}{N} \sum_{i=1}^{N} f(\psi_i), \quad \text{for all } f \in C(S, \mathbb{C}).$$

(This corresponds to choosing randomly, with equal probability, one of the vectors. Analytically, it is a mixture of Dirac measures: $\nu_F = \frac{1}{N}\sum_{i=1}^{N} \delta_{\psi_i}.$) Consider then $\mathcal{H} = \mathbb{C}^2$ and the two measures ν_1 and ν_2 defined by the collections $F_1 := \{(1, 0), (0, 1)\}$ and $F_2 := \{\psi_+, \psi_-\}$, where $\psi_\pm = \left(\frac{\mathrm{i}}{\sqrt{2}}, \pm\frac{\mathrm{i}}{\sqrt{2}}\right) \in \mathcal{H}$.

(c) Find the operators ρ_1 and ρ_2 corresponding to ν_1 and ν_2 explicitly.