Introduction to mathematical physics: Homework set 3
Quantum dynamics 10.2.2017

Exercise 1

Every $\phi, \psi \in \mathcal{H}$ satisfies a *polarization identity*:

$$(\phi, \psi) = \frac{1}{4}\left(\|\phi+\psi\|^2 - \|\phi-\psi\|^2 - \mathrm{i}\|\phi+\mathrm{i}\psi\|^2 + \mathrm{i}\|\phi-\mathrm{i}\psi\|^2\right) .$$

The identity can be generalized to include an action with a bounded operator: for all $\phi, \psi \in \mathcal{H}$ and $A \in \mathcal{B}(\mathcal{H})$

$$(\phi, A\psi) = \frac{1}{4}\sum_{n=0}^{3} \mathrm{i}^{-n}(\phi + \mathrm{i}^n\psi, A(\phi + \mathrm{i}^n\psi)) . \tag{1}$$

(The polarization identity clearly corresponds to the case $A = 1$.)

(a) Prove the *generalized polarization identity* in (1).

(b) Suppose $T \in \mathcal{B}(\mathcal{H})$. Show that, if $(\psi, T\psi) = 0$ for all $\psi \in \mathcal{H}$, then $T = 0$. (This implies that the "expectation values" of T determine T uniquely.)

(c) An operator $T : D \to \mathcal{H}$ is called *positive* if $(\psi, T\psi) \geq 0$ for all $\psi \in D$. Prove that, if $T \in \mathcal{B}(\mathcal{H})$ is positive, then it is self-adjoint.

Exercise 2

Let $M \subset \mathcal{H}$ be a subspace. Show that $(M^\perp)^\perp = \overline{M}$.

Exercise 3

Let $T, S \in \mathcal{B}(\mathcal{H})$, and $\alpha \in \mathbb{C}$ be arbitrary. Show that all of the following statements hold for the related adjoint operators.

(a) $(T + S)^* = T^* + S^*$

(b) $(\alpha T)^* = \alpha^* T^*$

(c) $(ST)^* = T^* S^*$ (notation: $ST = S \circ T$)

(d) $T^{**} = T$ (notation: $T^{**} = (T^*)^*$)

(e) $\|T^*T\| = \|T\|^2$

This proves that "$*$" is an involution on $\mathcal{B}(\mathcal{H})$ which makes it into a C^*-algebra.

(Please turn over)

Exercise 4

Suppose $U \in \mathcal{B}(\mathcal{H})$. Show that the following statements are equivalent:

(a) U is a unitary operator: $U^*U = 1 = UU^*$.

(b) $R(U) = \mathcal{H}$ and $(U\psi, U\phi) = (\psi, \phi)$ for all $\psi, \phi \in \mathcal{H}$. (As in the lecture notes, we use here the notation $R(U)$ for the range of the operator U.)

(c) $R(U) = \mathcal{H}$ and $\|U\psi\| = \|\psi\|$ for all $\psi \in \mathcal{H}$.

Exercise 5

Two Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ are said to be isomorphic, if there exists a *unitary map* between them (a map is unitary if it is linear, invertible, and preserves the scalar product). We denote this by $\mathcal{H}_1 \cong \mathcal{H}_2$. Show that

(a) $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2 \cong L^2(\mathbb{R}^3, \mathbb{C}^2) \cong L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$. (Here $L^2(\mathbb{R}^3, \mathbb{C}^2)$ denotes the L^2-space of *two-component wavefunctions*, $\psi : \mathbb{R}^3 \to \mathbb{C}^2$, with a scalar product $(\phi, \psi) = \int \mathrm{d}x \sum_{i=1}^2 \phi_i(x)^* \psi_i(x)$.)

(b) $L^2([0,1]^d) \otimes L^2([0,1]^{d'}) \cong L^2([0,1]^{d+d'})$ for any d, $d' \in \mathbb{N}_+$. (Hint: Fourier series.)