Introduction to mathematical physics: Homework set 2
Quantum dynamics 3.2.2017

In all of the exercises, $\mathcal{H}$ denotes a *complex* Hilbert space with a scalar product $(\cdot,\cdot)$. As in the lecture notes, "$\mathbb{1}$" denotes here the generic characteristic function: $\mathbb{1}_{\{P\}} = 1$ if P is true, and $\mathbb{1}_{\{P\}} = 0$ if P is false. For example, $\mathbb{1}_{\{x>0\}} = 1$, if $x > 0$, and it is equal to 0 otherwise.

Exercise 1

Let $T : \mathcal{H} \to \mathcal{H}$ be linear. Show that T is continuous if and only if it is bounded.

Exercise 2

Let $X = \mathbb{Z}^d$ and define

$$\ell_2(X) = \left\{\psi : X \to \mathbb{C} \,\middle|\, \sum_{x\in X} |\psi(x)|^2 < \infty\right\},$$

and for $\phi, \psi \in \ell_2(X)$ define

$$(\phi, \psi) = \sum_{x\in X} \phi(x)^* \psi(x)\,.$$

Show that $(\cdot,\cdot)$ yields a scalar product on $\ell_2(X)$ which makes $\ell_2(X)$ into a Hilbert space.

(Hint: Hölder's inequality and completeness of $\mathbb{C}$. If you get stuck with proving completeness of $\ell_2(X)$, any introductory book on functional analysis should be able to help you out.)

Exercise 3

Let m_1 denote the Lebesgue measure on $\mathbb{R}$ and $\mathcal{M}$ the associated σ-algebra. Define the map $\mu : \mathcal{M} \to [0,\infty]$ by setting $\mu(A) := \mathbb{1}_{\{0\in A\}} + m_1(A)$ for all $A \in \mathcal{M}$.

(a) Show that μ is a positive measure.
(b) Suppose that $f \in L^2(\mu)$. What can you say about the value of f at 1? How about at zero?

(This is not supposed to be a technically difficult exercise. The point is to check your understanding of the relevant concepts.)

(Please turn over!)

Exercise 4

Densely defined bounded operators: A mapping $A : D \to \mathcal{H}$ is called an *operator* if D is a linear subspace of $\mathcal{H}$ and A is linear. A is then called *densely defined* if D is dense, that is, if its closure satisfies $\overline{D} = \mathcal{H}$. An operator $A' : D' \to \mathcal{H}$ is called an *extension* of A if $D \subset D'$ and $A'\psi = A\psi$ for all $\psi \in D$. Equivalently, one can say that A' *extends* A, and denote it by $A \subset A'$.

Show that, if a densely defined operator $A : D \to \mathcal{H}$ is bounded, then there exists a continuous extension $\overline{A} : \mathcal{H} \to \mathcal{H}$. In addition, show that the extension is unique (if $A' : \mathcal{H} \to \mathcal{H}$ is continuous and extends A, then $A' = \overline{A}$), and that $\|\overline{A}\| = \|A\|$.

Exercise 5

Fubini's theorem

Consider the following complex functions defined on $\mathbb{R} \times \mathbb{R}$: for $x, y \in \mathbb{R}$ set

$$F(x,y) := \mathrm{e}^{\mathrm{i}2\pi(x-y)} \mathbb{1}_{\{x \geq 0, y \geq 0, 0 \leq x-y \leq 1\}}\,, \text{ and}$$
$$G(x,y) := F(x,y)(1+|y|^{3/2})^{-1}(1+\ln(1+|xy|))^{-1}\,.$$

(The second factor in F is the characteristic function of the set of those (x, y) for which $x \geq 0$, $y \geq 0$, and $0 \leq x - y \leq 1$.)

(a) Compute $I_1(x) := \int_{\mathbb{R}} \mathrm{d}y'\, F(x, y')$ and $I_2(y) := \int_{\mathbb{R}} \mathrm{d}x'\, F(x', y)$ for $x, y \in \mathbb{R}$. Conclude that $\int_{\mathbb{R}} \mathrm{d}x\, I_1(x) \neq \int_{\mathbb{R}} \mathrm{d}y\, I_2(y)$. Why is this not a contradiction to Fubini's Theorem?

(b) Show that $\int_{\mathbb{R}} \mathrm{d}x\, J_1(x) = \int_{\mathbb{R}} \mathrm{d}y\, J_2(y)$ for $J_1(x) := \int_{\mathbb{R}} \mathrm{d}y'\, G(x, y')$ and $J_2(y) := \int_{\mathbb{R}} \mathrm{d}x'\, G(x', y)$.