Introduction to mathematical physics: Quantum dynamics

Homework set 1
27.1.2017

These exercises are meant to recall basic properties of matrices and positive measures. The first exercise session will be on **Friday 27.1. at 12–14 in Exactum C321**.

Exercise 1

Let $M \in \mathbb{C}^{d\times d}$ be a complex matrix, with $d \in \mathbb{N}_+$ given. Show that the sum in

$$\mathrm{e}^M := \sum_{n=0}^{\infty} \frac{1}{n!} M^n$$

is absolutely convergent and its matrix norm satisfies $\|\mathrm{e}^M\| \le \mathrm{e}^{\|M\|}$. (Reminder: $M^0 = 1$ in such sums and the matrix norm satisfies $\|AB\| \le \|A\|\|B\|$ for any square matrices A and B.)

Exercise 2

Suppose that $M \in \mathbb{C}^{d\times d}$ is *diagonalizable*: there exists an invertible matrix $A \in \mathbb{C}^{d\times d}$ such that $\Lambda := A^{-1}MA$ is a diagonal matrix (i.e., $\Lambda_{ij} = 0$ if $i \ne j$). For every $t \in \mathbb{R}$ define a diagonal matrix $D_t \in \mathbb{C}^{d\times d}$ by setting $(D_t)_{ii} := \mathrm{e}^{t\Lambda_{ii}}$, for $i = 1, 2, \ldots, d$. Set then $E_t := AD_tA^{-1}$ for $t \in \mathbb{R}$.

Prove that $E_t = \mathrm{e}^{tM}$. (Hence, the two definitions of "matrix exponentiation" agree with each other.)

Exercise 3

Assume M and E_t, $t \in \mathbb{R}$, are given as in Exercise 2. Show that at any "time" $t \in \mathbb{R}$ and for any "initial data" $\psi \in \mathbb{C}^d$, we have

$$\lim_{\varepsilon\to 0} \left\| \frac{E_{t+\varepsilon}\psi - E_t\psi}{\varepsilon} - ME_t\psi \right\| = 0\,.$$

(This shows that the vector function $\psi_t := E_t\psi$ satisfies a differential equation $\partial_t\psi_t = M\psi_t$. So you just, hopefully, proved the existence (together with a beautiful representation) of a solution to a linear system of ODEs with constant coefficients!)

(Please turn over!)

Exercise 4

Let $f \in L^1(\mathbb{R})$ and $\varphi \in C_c^{(1)}(\mathbb{R})$. (In other words, assume that $|f|$ is integrable and φ is continuously differentiable with a compact support.) Consider the convolution function $g = \varphi * f$ defined by

$$g(t) := \int_{\mathbb{R}} \mathrm{d}x \, \varphi(t-x) f(x) \, , \quad t \in \mathbb{R} \, .$$

Show that g is continuously differentiable and that $\frac{\mathrm{d}}{\mathrm{d}t} g(t) = (\varphi' * f)(t) = \int_{\mathbb{R}} \mathrm{d}x \, \varphi'(t-x) f(x)$ for all $t \in \mathbb{R}$. As a corollary, conclude also that if φ above is smooth (it has derivatives of any order) then so is g. (Hint: Mean value theorem and Dominated convergence)

Exercise 5

Browse through the material in the first two chapters of Hall's book (chapters "The Experimental Origins of Quantum Mechanics" and "A First Approach to Classical Mechanics", pp. 1–46). The goal is to get familiar with the physical background and basic properties of classical and quantum mechanics.