6. Free evolution on $\mathbb{R}^d$: test-functions & Fourier-transforms

6.1. Multi-indices

... are a clever notation which lets one do analysis on $\mathbb{R}^d$ without drowning in an "index-soup".

Defn. Consider $\mathbb{R}^d$, for $d \geq 1$. A multi-index is a d-vector of non-negative integers, i.e. $\alpha \in \mathbb{N}_0^d$. It will be used ~~via~~ in the following definitions:

a) For $x \in \mathbb{R}^d$: $\quad x^\alpha := \prod_{i=1}^{d} x_i^{\alpha_i} \quad (\in \mathbb{R})$

b) For $f: \mathbb{R}^d \to \mathbb{C}$ and $x \in \mathbb{R}^d$:

$$(\partial^\alpha f)(x) := \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_d^{\alpha_d} f \Big|_x$$

c) $|\alpha| := \sum_{i=1}^{d} \alpha_i$ is called the order of α.

d) $\alpha \leq \beta$ means $\alpha_i \leq \beta_i \quad \forall i = 1, \dots, d$.

e) $\alpha - \beta = (\alpha_1 - \beta_1, \dots, \alpha_d - \beta_d) \in \mathbb{Z}^d$

f) $\alpha! := \alpha_1! \cdots \alpha_d!$

g) $\binom{\alpha}{\beta} := \dfrac{\alpha!}{\beta!\,(\alpha-\beta)!}$ for $\alpha \geq \beta$.

* Examples of uses of multi-indices

a) Taylor-expansions:

$$f(x) = \sum_{\alpha : |\alpha| \leq N-1} \frac{\partial^\alpha f(x_0)}{\alpha!} (x-x_0)^\alpha + \sum_{\alpha : |\alpha| = N} \frac{\partial^\alpha f(\xi)}{\alpha!} (x-x_0)^\alpha$$

b) Leibniz rules:

$$\partial^\alpha (fg) = \sum_{\beta : \beta \leq \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta} f \, \partial^\beta g$$

$$(x+y)^\alpha = \sum_{\beta : \beta \leq \alpha} \binom{\alpha}{\beta} x^{\alpha-\beta} y^\beta$$

6.2. The Schwartz space aka rapidly decreasing test-functions, $\mathcal{S}(\mathbb{R}^d)$

$$= \mathcal{S}_d := \{ f \in C^\infty(\mathbb{R}^d) \mid \|f\|_{\mathcal{S},N} < \infty \ \forall N = 0, 1, \ldots \}$$

where

$$\|f\|_{\mathcal{S},N} := \max_{\substack{\alpha, \beta \\ |\alpha|, |\beta| \leq N}} \sup_{x \in \mathbb{R}^d} |x^\alpha \partial^\beta f(x)| .$$

= { smooth functions which, along with all of their derivatives, decrease faster than any power at infinity }

* $\mathcal{S}$ is endowed with a metric: For $f, g \in \mathcal{S}$, let

$$d_{\mathcal{S}}(f,g) := \sum_{N=0}^\infty 2^{-N} \frac{\|f-g\|_{\mathcal{S},N}}{1 + \|f-g\|_{\mathcal{S},N}} \quad \left(\leq \sum_{N=0}^\infty 2^{-N} = 2 \right)$$

* The topology induced by d_S makes S into a <u>Fréchet space</u>: it is a topological vector space, topology defined by a <u>complete</u> invariant metric d_S (and it has a local base, whose elements are convex)

* Note that, if $f \in S_d$ and $P = \sum_{|\alpha| \le N} c_\alpha x^\alpha$ is an arbitrary polynomial, then

$$\int_{\mathbb{R}^d} dx \, |f(x) P(x)| < \infty \;, \quad \text{i.e. } fP \in L^1(\mathbb{R}^d),$$

since $|f(x) P(x)| \le \sum_{|\alpha| \le N} |c_\alpha| \, |x^\alpha f(x)|$

and for $|x| \ge 1$ we have

$$(1 + x^2)^n = (x^2)^n \left(1 + \frac{1}{x^2}\right)^n \le |x|^{2n} 2^n$$

where $|x|^{2n} = \left(\sum_{i=1}^d x_i^2\right)^n \le (d \max_i x_i^2)^n$

Therefore, $|x^\alpha f(x)| (1+x^2)^n \le (2d)^n \|f\|_{S, N+2n}$

$$\Rightarrow \int_{\mathbb{R}^d} dx \, |f(x) P(x)| \le \int dx \, (1+x^2)^{-n} \underbrace{(2d)^n \|f\|_{S, N+2n}}_{\times \sum_{|\alpha| \le N} |c_\alpha|}$$

$< \infty$ if $2n > d \Leftrightarrow n > \frac{d}{2}$.

6.3. Fourier transforms

Let us define, for $f \in S$, $\mathcal{F} f$ and $\tilde{\mathcal{F}} f$ by

$$(\mathcal{F} f)(k) := \int_{\mathbb{R}^d} dx \, e^{-i 2\pi x \cdot k} f(x) \qquad \forall k \in \mathbb{R}^d$$

$$(\tilde{\mathcal{F}} f)(y) := \int_{\mathbb{R}^d} dk \, e^{+i 2\pi y \cdot k} f(k) \qquad \forall y \in \mathbb{R}^d$$

$$= (\mathcal{F} f)(-y)$$

* Compared to usual definitions, we have included the 2π-factor in the exponent. This simplifies many of standard results (most notably: the Poisson resummation formula, convolutions, and relation to discrete Fourier-transform) the relation between the standard definition used in physics (p) and the one used here (k) is simply $p = 2\pi k$.

$$\int dx\, (\mathcal{F}f)(x)\, g(x) = \int dx\, f(x)\, (\mathcal{F}g)(x). \quad (*)$$

Theorem $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ and it is invertible, with $\mathcal{F}^{-1} = \tilde{\mathcal{F}}$. In addition, $\forall f, g \in \mathcal{S}$

Proof. (**) Let us first consider $G = \prod_{i=1}^{d} g_1(x_i)$, where $g_1(x) = e^{-\frac{1}{2}x^2}$. ($x \in \mathbb{R}$).

$$\Rightarrow G(x) = e^{-\frac{1}{2}x^2} \quad (x \in \mathbb{R}^d)$$

and $G \in \mathcal{S}$. Clearly,

$$(\mathcal{F}G)(k) = \int dx\, e^{-i2\pi x\cdot k} \prod_{i=1}^{d} g_1(x_i)$$

$$= \prod_{i=1}^{d} (\mathcal{F}_1 g_1)(k_i).$$

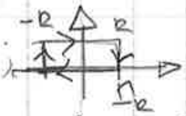
Here $(\mathcal{F}_1 g_1)(k) = \int_{-\infty}^{\infty} dx\, e^{-i2\pi xk} e^{-\frac{1}{2}x^2}$

and $\frac{1}{2}x^2 + i2\pi xk = \frac{1}{2}\left(x^2 + i4\pi xk + (i2\pi k)^2 - (i2\pi k)^2 \right)$

$$= \frac{1}{2}(x + i2\pi k)^2 + \frac{1}{2}(2\pi k)^2$$

$$\Rightarrow (\mathcal{F}_1 g_1)(k) = e^{-\frac{1}{2}(2\pi k)^2} \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}(x + i2\pi k)^2}$$

$$= e^{-\frac{1}{2}(2\pi k)^2} \lim_{R\to\infty} \int_{\substack{z = x + i2\pi k \\ x : -R \to R}} dz\, e^{-\frac{1}{2}z^2}$$



$\int_{\Gamma_R} dz\, e^{-\frac{1}{2}z^2} = 0$

$$\overset{\text{Cauchy}}{=} e^{-\frac{1}{2}(2\pi k)^2} \int_{-\infty}^{\infty} dy\, e^{-\frac{1}{2}y^2} = C e^{-\frac{1}{2}(2\pi k)^2}$$

and $C^2 = \int dy_1 dy_2\, e^{-\frac{1}{2}(y_1^2 + y_2^2)} = \int_0^\infty dr\, r\, 2\pi\, e^{-\frac{1}{2}r^2}$

$x = r^2$

$$\stackrel{?}{=} 2\pi \cdot \frac{1}{2} \int_0^\infty dx\, e^{-\frac{1}{2}x} = 2\pi \frac{1}{2} \Big/_0^\infty \frac{1}{-\frac{1}{2}} e^{-\frac{1}{2}x}$$

$$= 2\pi \quad \Rightarrow \quad c = \sqrt{2\pi}$$

$$\Rightarrow \quad (\mathcal{F}G)(k) = \prod_{i=1}^{d} \left[\sqrt{2\pi}\, e^{-\frac{1}{2}(2\pi k_i)^2} \right]$$

$$= (2\pi)^{\frac{d}{2}} e^{-\frac{1}{2}(2\pi)^2 k^2} = (2\pi)^{\frac{d}{2}} G(2\pi k) \in S.$$

$$\Rightarrow \tilde{\mathcal{F}}(\mathcal{F}G)(y) = \int_{\mathbb{R}^d} dk\, e^{i2\pi y\cdot k} (2\pi)^{\frac{d}{2}} G(2\pi k)$$

$$\stackrel{p=2\pi k}{=} (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} dp\, e^{iy\cdot p} G(p)$$

$$= (2\pi)^{-\frac{d}{2}} (\mathcal{F}G)\left(\frac{y}{2\pi}\right) = G(y), \quad \forall y \in \mathbb{R}^d.$$

Thus $\tilde{\mathcal{F}}(\mathcal{F}G) = G \quad (= \mathcal{F}(\tilde{\mathcal{F}}G))$

To prove $(*)$, note that since $f, g \in S$ $\Rightarrow f, g \in L^1(\mathbb{R}^d)$, Fubini's theorem implies that

$$\int dx\, (\mathcal{F}f)(x)\, g(x) = \int dx \left[\int dy\, e^{-i2\pi x\cdot y} f(y)\right] g(x)$$

$$= \int dy \int dx\, f(y)\, e^{-i2\pi x\cdot y} g(x) = \int dy\, f(y)\, (\mathcal{F}g)(y).$$

therefore, $(*)$ holds.

Let us then prove that $\forall f \in S$, $x \in \mathbb{R}^d$

$$f(x) = \int_{\mathbb{R}^d} dk\, e^{i2\pi k\cdot x} (\mathcal{F}f)(k).$$

Let $\varepsilon > 0$ be arbitrary, and define G_ε by
$G_\varepsilon(x) = (2\pi\varepsilon^2)^{-d/2}\, G\left(\frac{x}{\varepsilon}\right)$

$$\Rightarrow (\mathcal{F}G_\varepsilon)(k) = (2\pi\varepsilon^2)^{-d/2} \int dx\, e^{-i2\pi x\cdot k} G\left(\frac{x}{\varepsilon}\right)$$

$$\stackrel{y=\frac{x}{\varepsilon}}{=} (2\pi\cancel{\varepsilon^2})^{-d/2}\, \cancel{\varepsilon^d}\, (\mathcal{F}G)(\varepsilon k) = G(2\pi\varepsilon k)$$

$$\Rightarrow G_\varepsilon(x) = (2\pi\varepsilon^2)^{-d/2}\, G(2\pi\varepsilon' x)\Big|_{\varepsilon' = \frac{1}{2\pi\varepsilon}}$$

$$= (2\pi\varepsilon^2)^{-d/2} (\mathcal{F}G_{\varepsilon'})(x)$$

But for all $f \in S$

$$\int dy\, f(x-y)\, G_\varepsilon(y) = (2\pi)^{-d/2}\, \varepsilon^{-d} \int_{\mathbb{R}^d} dy\, f(x-y)\, G\!\left(\tfrac{y}{\varepsilon}\right)$$

$$\overset{y' = \frac{y}{\varepsilon}}{=} (2\pi)^{-d/2} \int_{\mathbb{R}^d} dy'\, f(x - \varepsilon y')\, G(y')$$

$$\xrightarrow[\varepsilon \to 0]{\text{DCT}} (2\pi)^{-d/2} \int dy'\, f(x)\, G(y') = f(x)\, (2\pi)^{-d/2} \cdot (\mathcal{F}G)(0)$$

$$= f(x)\, G(0) = f(x).$$

But $G_\varepsilon(y) = (2\pi\varepsilon^2)^{-d/2} (\mathcal{F}G_{\varepsilon'})(y)\,;\ \varepsilon' = \frac{1}{2\pi\varepsilon}$

Therefore, by $(*)$ and $\int dy\, e^{-i2\pi k\cdot y} f(x-y) = e^{-i2\pi k\cdot x} (\mathcal{F}f)(k)$

we thus have

$$\int dy\, f(x-y)\, G_\varepsilon(y) = (2\pi\varepsilon^2)^{-d/2} \int dy\, f(x-y) (\mathcal{F}G_{\varepsilon'})(y)$$

$$= (2\pi\varepsilon^2)^{-d/2} \int dk\, e^{-i2\pi k\cdot x} (\mathcal{F}f)(-k)\, G_{\varepsilon'}(k)$$

$$\overset{k' = -k}{=} (2\pi\varepsilon^2)^{-d/2} \int_{\mathbb{R}^d} dk'\, e^{i2\pi k'\cdot x} (\mathcal{F}f)(k') \cdot (2\pi\varepsilon'^2)^{-d/2} \cdot G(-2\pi\varepsilon k')$$

$$= \cancel{\left[2\pi\varepsilon^2 \cdot 2\pi \cdot \left(\frac{1}{2\pi\varepsilon}\right)^2 \right]^{-d/2}}$$

$$\times \int dk\, e^{i2\pi k\cdot x} (\mathcal{F}f)(k)\, e^{-\frac{1}{2}(2\pi\varepsilon k)^2}$$

$$\xrightarrow[\varepsilon \to 0]{\text{DCT}} \int dk\, e^{i2\pi k\cdot x} (\mathcal{F}f)(k)$$

Therefore, $\forall x \in \mathbb{R}^d:\ f(x) = \int dk\, e^{i2\pi x\cdot k} (\mathcal{F}f)(k).$

To prove that $\mathcal{F}f \in S$, note that (see also Exercise 6.?.) with $\hat{f} = \mathcal{F}f$,

$$k^\alpha \partial^\beta \hat{f}(k) = \int dx\, (-i2\pi x)^\beta e^{-i2\pi k\cdot x} \frac{1}{(i2\pi)^{|\alpha|}} \partial^\alpha f(x)$$

$(**)$ $\Rightarrow$ $\hat{f}$ smooth, and $\|\hat{f}\|_{S,N} \le C_d \|f\|_{S,N+d+1} < \infty$

$\Rightarrow$ $\hat{f} \in S$.

Therefore, $f = \tilde{\mathcal{F}}(\mathcal{F}f) \quad \forall f \in \mathcal{S}$,

and $\mathcal{F}f = 0 \Rightarrow f = 0$. Thus $\mathcal{F}$ is injective.

Since also $f(x) = (\mathcal{F}^2 f)(-x)$
$\Rightarrow f = \mathcal{F}^4 f$ and thus $\mathcal{F}^4 = \mathrm{id}_{\mathcal{S}}$.
Therefore, $\mathcal{F}$ is also onto.

$\Rightarrow \mathcal{F} : \mathcal{S} \to \mathcal{S}$ is invertible,
and then $\tilde{\mathcal{F}} = \mathcal{F}^{-1} = \mathcal{F}^3$. $\square$

Def. The convolution "$*$" is defined as

$$(f * g)(x) = \int_{\mathbb{R}^d} dy\, f(x-y)\, g(y), \quad x \in \mathbb{R}^d.$$

6.4 Proposition (properties of $\mathcal{F} : \mathcal{S} \to \mathcal{S}$)

a) $\forall f, g \in \mathcal{S} : \mathcal{F}(fg) = \mathcal{F}f * \mathcal{F}g$

b) $\forall f, g \in \mathcal{S} : f * g \in \mathcal{S}$ and

$$\mathcal{F}(f * g) = (\mathcal{F}f)(\mathcal{F}g)$$

c) The Parseval formula holds:

$$\forall f, g \in \mathcal{S} : \int dx\, f(x)^* g(x) = \int dk\, (\mathcal{F}f)(k)^* (\mathcal{F}g)(k)$$

d) $\forall f \in \mathcal{S} : \|\mathcal{F}f\|_2 = \|f\|_2 < \infty$.

Proof. Let $f, g \in \mathcal{S}$ be arbitrary, and denote $\hat{f} = \mathcal{F}f$, $\hat{g} = \mathcal{F}g$. By Fubini

$$\int dx\, e^{-i2\pi k \cdot x} \left[\int dy\, f(x-y)\, g(y) \right]$$
$$= \int dy\, g(y) \int dx\, e^{-i2\pi k \cdot (x-y+y)} f(x-y)$$
$$= (\mathcal{F}g)(k)(\mathcal{F}f)(k) \quad \forall k \in \mathbb{R}^d.$$

Applying this to $\hat{f}, \hat{g}$ instead of f, g shows that $\hat{f} * \hat{g} \in L^1$ and $\forall k$

$$\int dx\, e^{-i2\pi k\cdot x} (\hat{f} * \hat{g})(x) = (\mathcal{F}\hat{f})(k)(\mathcal{F}\hat{g})(k)$$
$$= f(-k)\, g(-k)$$

$\Rightarrow$

$$f(x)g(x) = \int dk'\, e^{+i2\pi k'\cdot x} (\hat{f} * \hat{g})(k')$$

Since $fg \in \mathcal{S}$

$\Rightarrow \forall x \in \mathbb{R}^d$: $0 = \int dk\, e^{i2\pi k\cdot x}$

$$\left[\mathcal{F}(fg)(k) - (\hat{f} * \hat{g})(k) \right]$$

By Fubini, then $\forall \phi \in \mathcal{S}$

$$0 = \int dx\, \phi(x) \left(\int dk\, e^{+i2\pi k\cdot x} [\,] \right)$$

$$= \int dk\, \hat{\phi}(-k) \left[\mathcal{F}(fg)(k) - (\hat{f} * \hat{g})(k) \right]$$

But since $\mathcal{F}$ is invertible $\Rightarrow$

$$\forall \phi \in \mathcal{S}: \quad \int dk\, \phi(k) [\,] = 0$$

$\Rightarrow$ $(\hat{f} * \hat{g})(k) = \mathcal{F}(fg)(k)$ a.e. $k \in \mathbb{R}^d$.

But as $\hat{f} * \hat{g}$ is continuous (use DCT), we have that $\hat{f} * \hat{g} = \mathcal{F}(fg)$ pointwise
$\Rightarrow$ a) holds.
But then $\hat{f} * \hat{g} \in \mathcal{S}$, $\forall f, g \in \mathcal{S}$
and $\mathcal{F}^{-1}(\hat{f} * \hat{g}) = fg$, $\Rightarrow \mathcal{F}(\hat{f} * \hat{g})(-k) = f(k)g(k)$
Applying this for $\mathcal{F}^{-1} f$, $\mathcal{F}^{-1} g$
$\Rightarrow \forall k$: $\mathcal{F}(f * g)(k) = (\mathcal{F}^{-1} f)(-k)(\mathcal{F}^{-1} g)(-k)$
$= \hat{f}(k)\hat{g}(k)$
Thus b) holds, as well.

By (*) in Thm. 6.3, we have

$$\int dx\, f(x)^* g(x) = \int dx\, \mathcal{F}(\mathcal{F}^{-1} f^*)(x)\, g(x)$$

$$= \int dk\, (\mathcal{F}^{-1} f^*)(k)\, (\mathcal{F} g)(k)$$

where $(\mathcal{F}^{-1}f^*)(k) = (\mathcal{F}f^*)(-k)$

$$= \int dx\, e^{-i2\pi(-k)\cdot x} f(x)^*$$

$$= \left[\int dx\, e^{-i2\pi k\cdot x} f(x)\right]^* = (\mathcal{F}f)(x)^*$$

Thus Parseval holds. Since $|f|^2 = f^* f \in \mathcal{S} \Rightarrow |f|^2 \in L^1 \Rightarrow f \in L^2 \;\; \forall f \in \mathcal{S}$.
We have, in particular,

$$\int dx\, f(x)^* f(x) = \int dk\, \hat{f}(k)^* \hat{f}(k) < \infty$$

$$\Rightarrow \|f\|_2 = \|\mathcal{F}f\|_2 < \infty \quad \square$$

6.5. Fourier transform on $L^2(\mathbb{R}^d)$

Since $C_c^\infty(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$, and C_c^∞ is dense in L^2, also $\mathcal{S}_d$ is dense in $L^2(\mathbb{R}^d)$. $\mathcal{S}_d$ is thus a dense linear subspace of $L^2(\mathbb{R}^d)$, and by 6.4. d) $\mathcal{F}: \mathcal{S} \to \mathcal{S}$ is a linear isometry on $L^2(\mathbb{R}^d)$ with a domain $D(\mathcal{F}) = \mathcal{S}$.

Therefore, applying the results proven in Exercise 7.2., there is a unique continuous extension $\overline{\mathcal{F}}: \overline{\mathcal{S}} \to \overline{\mathcal{S}}$ which is also an isometry. Since $\overline{\mathcal{S}} = L^2(\mathbb{R}^d)$, now Exercise 2.4. implies that $\overline{\mathcal{F}}: L^2 \to L^2$ is actually a unitary operator.

Definition: The unique extension $\overline{\mathcal{F}}: L^2 \to L^2$ of $\mathcal{F}: \mathcal{S} \to \mathcal{S}$ is called Fourier transform on $L^2(\mathbb{R}^d)$.

From now on, we will denote also $\overline{\mathcal{F}}: L^2 \to L^2$ by $\mathcal{F}$. If distinction needs to be made, we use $\mathcal{F}_\mathcal{S}$ and $\mathcal{F}_{L^2}$. (Note: Pointwise invertibility is true only for $\mathcal{F}_\mathcal{S}$.)

Proposition: a) $\mathcal{F}_{L^2}$ is a unitary operator, and $\mathcal{F}_{L^2}^* = \mathcal{F}_{L^2}^{-1} = \overline{\mathcal{F}_S^{-1}} = \mathcal{F}_{L^2}^3$.

Proof: We already proved unitarity.
Since $\mathcal{F}_S^{-1} = \tilde{\mathcal{F}}_S$ also is an isometry, $\tilde{\mathcal{F}}_{L^2} := \overline{\mathcal{F}_S^{-1}}$ is a unitary operator.

Since for all $\psi \in S$ then $\tilde{\mathcal{F}}_{L^2} \mathcal{F}_{L^2} \psi$
$= \tilde{\mathcal{F}}_{L^2}(\mathcal{F}_S \psi) = \mathcal{F}_S^{-1}(\mathcal{F}_S \psi) = \psi$.
Also $\mathcal{F}_{L^2}^4 \psi = \mathcal{F}_S^4 \psi = \psi$.
It follows that $\tilde{\mathcal{F}}_{L^2} = \mathcal{F}_{L^2}^{-1} = \mathcal{F}_L^* = \mathcal{F}_{L^2}^3$ □

Remarks: * Note that $\mathcal{F}_{L^2}$ unitary implies Parseval formula:

$$\forall \psi, \phi \in L^2 : (\mathcal{F}\psi, \mathcal{F}\phi) = (\psi, \phi).$$

* There is no representation of $\mathcal{F}_{L^2}$ as an integral operator. (It is, however, the unique contin. extension of the integral operator with integral kernel $K(x,y) = e^{-i2\pi x\cdot y}$.)
However, the following formulae hold:

a) If $\psi \in L^1 \cap L^2$, then for all $f \in S$:

$$(f, \mathcal{F}\psi) = (\mathcal{F}_{L^2}^{-1} f, \psi) = (\tilde{\mathcal{F}}_S f, \psi)$$

$$= \int dx\, \psi(x) \left[\int dk\, e^{i2\pi x\cdot k} f(k) \right]^*$$

$$\overset{\text{Fubini}}{=} \int dk\, f(k)^* \left[\int dx\, e^{-i2\pi x\cdot k} \psi(x) \right]$$

$$\Rightarrow \quad (\mathcal{F}\psi)(k) = \int dx\, e^{-i2\pi k\cdot x} \psi(x), \text{ a.e. } k \in \mathbb{R}^d.$$

b) If $\psi \in L^2 \setminus L^1$, then $\psi_R(x) := \mathbb{1}(|x| \le R)\psi(x)$
$\Rightarrow \psi_R \in L^1 \cap L^2$. (Hölder: $\int dx\, |\psi_R| \le \|\psi\| \sqrt{\int_{|x|\le R} dx}$)
and $\psi_R \xrightarrow{R\to\infty} \psi$ in L^2-norm.

... Thus there is a sequence $R_n \to \infty$ such that for a.e. $k \in \mathbb{R}^d$

$$(\mathcal{F}\psi)(k) = \lim_{n\to\infty} \int_{|x|\le R_n} dx\, e^{-i2\pi x\cdot k} \psi(x).$$

6.6. Differential operators on $L^2(\mathbb{R}^d)$

1. Lemma Suppose A is a closed, densely defined operator, and U is a unitary operator. Then AU and UA are closed and densely defined. In addition,

$$(AU)^* = U^*A^*, \quad (UA)^* = A^*U^*.$$

Proof. The products are defined with their natural domains:

$$D(AU) = \{\psi \in \mathcal{H} \mid U\psi \in D(A)\} = U^* D(A)$$

$$D(UA) = D(A).$$

For any unitary map $\mathcal{U} : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$ and subset $S \subset \tilde{\mathcal{H}}$ we have $\overline{\mathcal{U}S} = \mathcal{U}\overline{S}$. Thus both $D(A)$ and $U^*D(A)$ are dense. Now

$$\mathcal{G}(AU) = \{(\psi, AU\psi) \mid U\psi \in D(A)\} = \{(U^*\phi, A\phi) \mid \phi \in D(A)\} = \mathcal{U}_1 \mathcal{G}(A) \text{ with}$$

$$\mathcal{U}_1(\psi,\phi) := (U^*\psi, \phi).$$

Clearly, $\mathcal{U}_1 : \mathcal{H}\oplus\mathcal{H} \circlearrowleft$ is unitary, and thus $\overline{\mathcal{G}(AU)} = \mathcal{U}_1 \overline{\mathcal{G}(A)} = \mathcal{U}_1 \mathcal{G}(A) = \mathcal{G}(AU)$ and AU is closed and densely def.

Similarly, $\mathcal{G}(UA) = \mathcal{U}_2 \mathcal{G}(A)$, with $\mathcal{U}_2((\psi,\phi)) := ((\psi, U\phi))$ which is unitary. Thus also UA is closed and densely def.

Suppose $\tilde{\phi}, \phi \in \mathcal{H}$. Then $\phi \in D((AU)^*)$, $\tilde{\phi} = (AU)^*\phi$

$\Leftrightarrow \forall \psi \in D(AU) : (\phi, AU\psi) = (\tilde{\phi}, \psi) = (U\tilde{\phi}, U\psi)$

$\Leftrightarrow \forall \varphi \in D(A) : (\phi, A\varphi) = (U\tilde{\phi}, \varphi)$

$\Leftrightarrow \phi \in D(A^*)$ and $U\tilde{\phi} = A^*\phi$

$\Leftrightarrow \quad \phi \in D(A^*) = D(U^*A^*)$ and $\tilde{\phi} = U^*A^*\phi$.

This proves that $(AU)^* = U^*A^*$.
Similarly, $\phi \in D((UA)^*)$, $\tilde{\phi} = (UA)^*\phi$
$\Leftrightarrow \forall \psi \in D(UA): \ (\tilde{\phi}, \psi) = (\phi, UA\psi) = (U^*\phi, A\psi)$
$\Leftrightarrow \forall \psi \in D(A): \ (\tilde{\phi}, \psi) = (U^*\phi, A\psi)$
$\Leftrightarrow \ U^*\phi \in D(A^*)$ and $\tilde{\phi} = A^*(U^*\phi)$
$\Leftrightarrow \ \phi \in U D(A^*) = (U^*)^* D(A^*) = D(A^*U^*)$
and $\tilde{\phi} = A^*U^*\phi$
and thus $(UA)^* = A^*U^*$ $\square$

2. Corollary If A is self-adjoint and U is unitary, then U^*AU and UAU^* are self-adjoint.

3. Definition: Let α be a multi-index. Then $V_\alpha(k) = (i2\pi k)^\alpha$ defines a closed multiplication operator M_{V_α} on $L^2(\mathbb{R}^d)$.
We define the operator ∂^α on $L^2(\mathbb{R}^d)$ by the formula $\partial^\alpha := \mathcal{F}_{L^2}^{-1} M_{V_\alpha} \mathcal{F}_{L^2}$.

4. Properties: a) ∂^α is a closed, densely defined operator.
b) $A := (-i\partial)^\alpha$ is self-adjoint. If α_i is even $\forall i$, then A is also positive. ($\Leftrightarrow$ A self-adj. and $\sigma(A) \subset [0,\infty)$)
c) $\forall f \in \mathcal{S}: \ \partial^\alpha_{L^2} f = \partial^\alpha f$ and, moreover, $\partial^\alpha_{L^2} = \overline{\partial^\alpha_{\mathcal{S}}}$.

d) "Partial integration" is possible:
$\forall \phi, \psi \in D(\partial^\alpha)$
$$(\phi, \partial^\alpha \psi) = ((-\partial)^\alpha \phi, \psi).$$

Proof: By Lemma 6.6.1. above, and unitarity of $\mathcal{F}_{L^2}$, "a)" follows since M_{V_α} is closed and densely defined (Exercise 5.1.) "b)" is a consequence of the corresponding properties of M_{V_α} and Lemma 6.6.7. on page 77b.

Now $(-i)^{|\alpha|} V_\alpha(k) = (2\pi k)^\alpha \in \mathbb{R} \quad \forall k \in \mathbb{R}^d \overset{!}{\Rightarrow} M_{(-i)^{|\alpha|} V_\alpha}$ self-adjoint

Corollary 6.6.2.
$\Rightarrow \quad A := \mathcal{F}^* M_{(-i)^{|\alpha|} V_\alpha} \mathcal{F} = (-i\partial)^\alpha$ is self-adjoint.

If also $\quad \alpha_i = 2n_i \ \forall i, \quad n \in \mathbb{N}_0^d,$
then

$$(\psi, A\psi) = (\psi, (-i\partial)^\alpha \psi) = (\hat{\psi}, (-i)^{|\alpha|} M_{V_\alpha} \hat{\psi})$$
$$= \int dk \, |\hat{\psi}(k)|^2 \, (2\pi)^{|\alpha|} \prod_{j=1}^{d} k_j^{2n_j} \geq 0 \quad \forall \psi \in D(A)$$

Lemma 6.6.7.
$\Rightarrow \quad \sigma(A) \subset [0, \infty).$

To prove "c)", assume $f \in \mathcal{S}$. Then

$$(M_{V_\alpha} \mathcal{F}_{L^2} f)(k) = (i2\pi k)^\alpha \hat{f}(k) \overset{\text{Exercise 7.4b)}}{=} \mathcal{F}_{\mathcal{S}}(\partial^\alpha f)(k) \in \mathcal{S}$$

$$\Rightarrow \partial^\alpha_{L^2} f = \mathcal{F}_{L^2}^{-1} M_{V_\alpha} \mathcal{F}_{L^2} f = \mathcal{F}_{\mathcal{S}}^{-1}(\mathcal{F}_{\mathcal{S}}(\partial^\alpha f)) = \partial^\alpha f.$$

Thus $\partial^\alpha_{L^2}$ is a closed extension of the densely defined operator $\partial^\alpha_{\mathcal{S}} : \mathcal{S} \to L^2$
$\Rightarrow \partial^\alpha_{\mathcal{S}}$ is closable, and $\overline{\partial^\alpha_{\mathcal{S}}} \subset \partial^\alpha_{L^2}$.

To show the converse, suppose $\psi \in D(\partial^\alpha_{L^2})$.
Then $\hat{\psi} \in D(M_{V_\alpha}) \Rightarrow V_\alpha \hat{\psi} \in L^2$.
Since also $\hat{\psi} \in L^2 \Rightarrow \exists g_n \in \mathcal{S}, n \in \mathbb{N}$, s.t. $g_n \to \hat{\psi}$ in norm. Let us then define $f_n = G_n g_n$ where

$$G_n(k) := e^{-\frac{1}{2}\varepsilon_n^2 k^{2\alpha}}, \quad \varepsilon_n = \|\hat{\psi} - g_n\|^{1/2}.$$

Clearly, also $f_n \in \mathcal{S} \ \forall n$, and $\varepsilon_n \xrightarrow{n\to\infty} 0$.
In addition,

$$\|\hat{\psi} - f_n\| = \|(1-G_n)\hat{\psi} + G_n(\hat{\psi} - g_n)\|$$
$$\leq \|(1-G_n)\hat{\psi}\| + \|G_n(\hat{\psi} - g_n)\|$$
$\to 0$ by easy applications of DCT.

Also $\|V^\alpha \hat{\psi} - V^\alpha f_n\| \leq \|(1-G_n) V^\alpha \hat{\psi}\| + \|G_n V^\alpha (\hat{\psi} - g_n)\| \to 0$
since

$$\|G_n V^\alpha (\hat{\psi} - g_n)\|^2 \leq \int dk \, e^{-\varepsilon_n^2 k^{2\alpha}} (2\pi)^{2|\alpha|} k^{2\alpha} \times |\hat{\psi}(k) - g_n(k)|^2$$
$$\leq (2\pi)^{2|\alpha|} \frac{1}{\varepsilon_n^2} \int dk \, |\hat{\psi}(k) - g_n(k)|^2 \leq (2\pi)^{2|\alpha|} \|\hat{\psi} - g_n\|$$
$\to 0.$

where we have applied the inequality
$x e^{-x} \leq e^{-1} < 1 \;\forall\, x \geq 0$.

Thus $f_n \to \hat{\psi}$ and $V_\alpha f_n \to V_\alpha \hat{\psi}$ in norm, with $f_n \in S$ $\forall n$. Let

$$\varphi_n = \mathcal{F}_S^{-1} f_n \in S. \Rightarrow \varphi_n \to \psi \text{ in } L^2 \text{ and}$$

$$\text{also } \partial^\alpha \varphi_n = \mathcal{F}_S^{-1}(V_\alpha f_n) \to \mathcal{F}_{L^2}^{-1}(M_{V_\alpha} \hat{\psi}) = \partial^\alpha \psi.$$

$$\text{Thus } (\psi, \partial^\alpha \psi) \in \overline{\mathcal{G}(\partial^\alpha|_S)} \overset{\text{Thrm 5.5.}}{=} \mathcal{G}(\overline{\partial^\alpha|_S}) \;\forall \psi \in D(\partial^\alpha)$$

Hence, $\mathcal{G}(\partial^\alpha|_{L^2}) = \mathcal{G}(\overline{\partial^\alpha|_S}) \Rightarrow \partial^\alpha|_{L^2} = \overline{\partial^\alpha|_S}$.

"d)" follows from

$$(\phi, \partial^\alpha \psi) = (\mathcal{F}\phi, \mathcal{F}\partial^\alpha \psi) = (\hat{\phi}, M_{V_\alpha} \hat{\psi})$$

$$= \int dk\, \hat{\phi}(k)^* (i 2\pi k)^\alpha \hat{\psi}(k)$$
$$= (-1)^{|\alpha|} \int dk\, ((i 2\pi k)^\alpha \hat{\phi}(k))^* \hat{\psi}(k)$$
$$= (-1)^{|\alpha|} (M_{V_\alpha} \hat{\phi}, \hat{\psi}) = (-1)^{|\alpha|} (\partial^\alpha \phi, \psi) \quad \square$$

5. <u>Definition</u> Let $F: \mathbb{R}^d \to \mathbb{C}$ be Lebesgue measurable. We define then

$$F(-i\partial) := \mathcal{F}_{L^2}^{-1} M_{V_F} \mathcal{F}_{L^2}$$

where $V_F(k) := F(2\pi k)$.

6. <u>Properties</u> a) $F(-i\partial)$ is a closed, densely defined operator.
b) If F is real, $F(-i\partial)$ is self-adjoint.
c) If F is positive, $F(-i\partial)$ is positive.

<u>Proof</u> Exactly as in 4. above. $\square$

7. Lemma: Suppose A is a normal operator on $\mathcal{H}$. Then

$$A \text{ is positive} \overset{①}{\Longleftrightarrow} \sigma(A) \subset [0,\infty[\overset{②}{\Longleftrightarrow} \langle \psi | A\psi \rangle \geq 0 \quad \forall \psi \in D(A)$$

Proof. "$\overset{①}{\Rightarrow}$" Follows from definition

"$\overset{①}{\Leftarrow}$" Let E be the PVM obtained from spectral decomposition of A. Then, by Theorem 5.22., for the function $f(\lambda) = \lambda$, $\lambda \in \sigma(A)$, we have $f(A) = A$, and thus by 5.24.b) $A^* = \mathcal{O}(f)^* = \mathcal{O}(f^*) = \mathcal{O}(f) = A$, since $\lambda^* = \lambda \;\forall \lambda \geq 0$. Thus $\sigma(A) \subset [0,\infty[\Rightarrow A$ is self-adjoint.

"$\overset{②}{\Rightarrow}$" By Thrm. 5.22, with E = PVM from the spectral decomposition, $\forall \psi \in D(A)$:

$$\langle \psi | A\psi \rangle = \int_{\sigma(A)} E_{\psi,\psi}(d\lambda)\, \lambda \geq 0,$$

since $E_{\psi,\psi}$ is a positive measure (5.20.a) and $\lambda \in \sigma(A) \Rightarrow \lambda \geq 0$.

"$\overset{②}{\Leftarrow}$" Assume $\langle \psi | A\psi \rangle \geq 0 \;\forall \psi \in D(A)$, and suppose, to get a contradiction, that $\exists \lambda_0 \in \sigma(A) \setminus [0,\infty[$. Denote $r := \mathrm{dist}(\lambda_0, [0,\infty[) > 0$, and set $\varepsilon := r/2 > 0$, $B := B_\varepsilon(\lambda_0) \cap \sigma(A)$, $P := E(B)$. Then $P \neq 0$, since otherwise the bounded operator $f(A)$ for $f(\lambda) := (\lambda - \lambda_0)^{-1} \mathbb{1}_{\{|\lambda - \lambda_0| \geq \varepsilon\}}$ would be an inverse of $A - \lambda_0 \Rightarrow \lambda_0 \notin \sigma(A)$ ↯. Thus we can choose $\psi \in R(P)$ with $\|\psi\| = 1$. By definition (items 5.19. "c)" and "a)"), then for the complement $B^c := \sigma(A) \setminus B$ the corresponding projection annihilates ψ, since $E(B^c)\psi = E(B^c)P\psi$
$= E(B^c)E(B)\psi = E(B^c \cap B)\psi = E(\emptyset)\psi = 0\psi = 0$.
Thus $E_{\psi,\psi}(B^c) := \langle \psi | E(B^c)\psi \rangle = \langle \psi | 0 \rangle = 0$.

$\Rightarrow \int_{\sigma(A)} E_{\psi,\psi}(d\lambda)\, g(\lambda) = \int_B E_{\psi,\psi}(d\lambda)\, g(\lambda)$ for any meas. g.

In particular,

$$\int_{\sigma(A)} E_{\psi,\psi}(d\lambda)\,|\lambda|^2 = \int_B E_{\psi,\psi}(d\lambda)\,\overbrace{|\lambda|^2}^{=|\lambda-\lambda_0+\lambda_0|^2}$$

$$\le \int_B E_{\psi,\psi}(d\lambda)\,(\varepsilon+|\lambda_0|)^2 \le (|\lambda_0|+\varepsilon)^2\,\|\psi\|^2 < \infty.$$

$\Longrightarrow$ $\psi \in D(A)$. (By 5.22. and 5.24)

Denote $z := \langle \psi | A\psi \rangle = \int_{\sigma(A)} E_{\psi,\psi}(d\lambda)\,\lambda$.

Then $z - \lambda_0 = z - \lambda_0 \|\psi\|^2 = \int_{\sigma(A)} E_{\psi,\psi}(d\lambda)\,(\lambda-\lambda_0)$

$$= \int_B E_{\psi,\psi}(d\lambda)\,(\lambda-\lambda_0)$$

$$\Longrightarrow |z-\lambda_0| \le \int_B E_{\psi,\psi}(d\lambda)\,\overbrace{|\lambda-\lambda_0|}^{<\varepsilon}$$

$$\le \varepsilon \int_B E_{\psi,\psi}(d\lambda) \le \varepsilon\|\psi\|^2 = \varepsilon = r/2 < r.$$

Thus $z \notin [0,\infty[$, since "$z\in[0,\infty[$" would require $|z-\lambda_0| \ge r$. This is a contradiction, and hence we can conclude that $\sigma(A) \subset [0,\infty[$. □

* "Faster" version of "$\overset{(2)}{\Longleftarrow}$": Suppose $\lambda_0 \in \sigma(A)\setminus[0,\infty[$. Then there is an approximate eigenvector $\psi \in D(A)$ for which $\|\psi\| = 1$ and $\|A\psi - \lambda_0\psi\| \le \varepsilon$, using $\varepsilon := \frac{r}{2}$, $r := \mathrm{dist}(\lambda_0, [0,\infty[) > 0$. (See p. 56)

But then $\langle\psi|A\psi\rangle - \lambda_0 = \langle\psi|A\psi\rangle - \lambda_0\|\psi\|^2$
$= \langle\psi|A\psi - \lambda_0\psi\rangle$

$\Longrightarrow |\langle\psi|A\psi\rangle - \lambda_0| \le \|\psi\|\,\|A\psi-\lambda_0\psi\| = \|A\psi-\lambda_0\psi\| \le \varepsilon < r.$

$\Longrightarrow \langle\psi|A\psi\rangle \notin [0,\infty[$ ↯ □

8. Corollary: If T is a normal operator, and $f: \sigma(T) \to [0,\infty[$ is Borel measurable, then $f(T)$ is a positive operator.

Proof: By 5.24., $f(T)$ is a normal operator, and if $\psi\in D(T)$,
$\langle\psi|f(T)\psi\rangle = \int_{\sigma(T)} E_{\psi,\psi}(d\lambda)\,f(\lambda) \ge 0$. □

6.7. Free Schrödinger evolution on $\mathbb{R}^d$

Consider the following operators on $L^2(\mathbb{R}^d)$

$$\hat{p}_j := -i\partial_j \, , \quad j = 1, \dots, d$$

$$\widehat{p_j^2} := -\partial_j^2 \, , \quad j = 1, \dots, d$$

$$H_0 := -\frac{1}{2}\Delta = \frac{1}{2}\sum_{j=1}^{d} (-i\partial_j)^2$$

By 6.6.4. and 6.6.6. all are densely defined and self-adjoint.

1. <u>Definition</u> For $t \in \mathbb{R}$, let $U_t = F_t(-i\partial)$ with $F_t(p) = e^{-it\frac{1}{2}p^2}$, defined as in 6.6.5. Then $\psi(t) := U_t \psi(0)$ defines the <u>free Schrödinger evolution</u> on $L^2(\mathbb{R}^d)$.

2. <u>Properties</u>: * Now $U_t = \mathcal{F}^{-1} M_{u_t} \mathcal{F}$ with

$$u_t(k) = e^{-it\frac{1}{2}(2\pi k)^2} = e^{-it\frac{(2\pi)^2}{2}k^2}.$$

Exercise 5.5. $\Rightarrow$ $(M_{u_t})_t$ is a strongly continuous unitary group, with generator M_V,

$V(k) = \frac{1}{2}(2\pi)^2 k^2$. Exercise 7.1. $\Rightarrow$ U_t is also a strongly continuous unitary group, with generator $A = \mathcal{F}^{-1} M_V \mathcal{F}_d$. Since $V(k) = \frac{p^2}{2}\Big|_{p=2\pi k}$

$$\Rightarrow A = \frac{1}{2}\sum_{j=1}^{d} (-i\partial_j)^2 = -\frac{1}{2}\Delta = H_0 .$$

Thus $U_t = e^{-itH_0}$ with $H_0 = -\frac{1}{2}\Delta = -\frac{1}{2}\nabla^2$.

* In general, the natural domains of combinations of unbounded operators are not very useful. Here, however, we do have

$$H_0 = \frac{1}{2}\sum_{j=1}^{d} \widehat{p_j^2} \quad \text{and} \quad \widehat{p_j^2} \neq \hat{p}_j \hat{p}_j .$$

3. Proposition: a) If $\psi(t_0) \in \mathcal{S}$ for some t_0, then $\psi(t) \in \mathcal{S}$ $\forall t$, and $\forall t \neq 0$

$$\psi(x,t) := \psi(t)(x) = \int_{\mathbb{R}^d} dy\, \mathcal{K}(x,y;t)\, \psi(y,0)$$

where $\mathcal{K}$ is called the free propagator and is given by

$$\mathcal{K}(x,y;t) = \frac{1}{(i2\pi t)^{d/2}}\, e^{i\frac{1}{2t}(x-y)^2}$$

In addition, $\psi(x,t)$ solves the free Schrödinger differential equation:

$$i\partial_t \psi(x,t) = -\frac{1}{2}\Delta_x \psi(x,t) \quad \forall t \in \mathbb{R}, x \in \mathbb{R}^d.$$

b) If $\psi(0) \in L^1 \cap L^2$, $t \neq 0$, then

$$\psi(x,t) = \int_{\mathbb{R}^d} dy\, \mathcal{K}(x,y;t)\psi(y,0) \quad \text{a.e. } x \in \mathbb{R}^d$$

c) In general, $\forall \psi \in L^2$, $t \neq 0$, $\exists R_n > 0$, $n \in \mathbb{N}$, s.t. $R_n \to \infty$ and for a.e. $x \in \mathbb{R}^d$

$$\psi(x,t) = \lim_{n\to\infty} \int_{|y| \le R_n} dy\, \mathcal{K}(x,y;t)\, \psi(y,0).$$

Proof. a) Assume $\psi(t_0) \in \mathcal{S}$, $t_0 \in \mathbb{R}$. By the group property of U_t, then $\forall t$

$$\psi(t) = U_t \psi(0) = U_{t-t_0} U_{t_0} \psi(0) = U_{t-t_0}\psi(t_0).$$

$$= \mathcal{F}_{L^2}^{-1} M_{u_{t-t_0}} \mathcal{F}_{L^2} \psi(t_0) = \mathcal{F}_{\mathcal{S}}^{-1} M_{u_{t-t_0}} \mathcal{F}_{\mathcal{S}} \psi(t_0)$$

$= \tilde{U}_{t-t_0}\psi(t_0)$, where $\tilde{U}_t$ denotes

the map "U_t" defined in Ex. 7.5.
Since then $\tilde{U}_{t-t_0}\psi(t_0) \in S$, we find that $\psi(t) \in S$ $\forall t \in \mathbb{R}$. But then also $\psi(0) \in S$

$\Rightarrow$ $\psi(t) = U_t \psi(0) = \tilde{U}_t \psi(0)$. Thus $\forall t \neq 0$, $x \in \mathbb{R}^d$,

$$\Rightarrow \quad \psi(x,t) = \int_{\mathbb{R}^d} dy\, \underbrace{K(x-y,t)}_{= \mathcal{K}(x,y;t)} \psi(y,0).$$

This proves the first part of "a)".

By the inversion formula, we have for all $t, \varepsilon \in \mathbb{R}$, $x \in \mathbb{R}^d$,

$$\psi(x,t+\varepsilon) - \psi(x,t) = \int_{\mathbb{R}^d} dk\, e^{i2\pi x\cdot k}\left[\hat{\psi}(k,t+\varepsilon) - \hat{\psi}(k,t)\right]$$

Since $\hat{\psi}(k,t) = e^{-it\frac{1}{2}(2\pi k)^2}\hat{\psi}(k,0)$

$$\Rightarrow \quad \psi(x,t+\varepsilon) - \psi(x,t) = \int dk\, e^{i2\pi x\cdot k}\left[e^{-i\varepsilon\frac{1}{2}(2\pi k)^2} - 1\right]\hat{\psi}(k,t)$$

Since $\left| e^{-i\varepsilon\frac{1}{2}(2\pi k)^2} - 1 \right| \leq |\varepsilon| \frac{1}{2}(2\pi)^2 k^2$

and $k^2 \hat{\psi}(k,t) \in L^1$, DCT

$$\Rightarrow \quad \lim_{\varepsilon \to 0} \frac{\psi(x,t+\varepsilon) - \psi(x,t)}{\varepsilon} = \int dk\, e^{i2\pi x\cdot k}\hat{\psi}(k,t) \times \underbrace{\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[e^{-i\varepsilon\frac{1}{2}(2\pi k)^2} - 1\right]}_{= -i\frac{1}{2}(2\pi k)^2 = -i\left[-\frac{1}{2}(i2\pi k)^2\right]}$$

$$\Rightarrow \quad i\partial_t \psi(x,t) = -\frac{1}{2}\int dk\, e^{i2\pi x\cdot k}\sum_{j=1}^{d}(i2\pi k_j)^2\hat{\psi}(k,t)$$

$$= -\frac{1}{2}\nabla_x^2 \psi(x,t) \quad \text{by Exercise 7.4b)}$$

This completes the proof of "a)".

For "b)", Let $\psi(0) \in L^1 \cap L^2$, $t \neq 0$.

Then $\forall f \in \mathcal{S}$: $(f, \psi(t)) = (f, U_t \psi(0))$

$= (U_t^* f, \psi(0)) = (U_{-t} f, \psi(0))$

$= \int dx\, \psi(x,0) \left[\int dy\, K(x,y;-t)\, f(y) \right]^*$

Fubini

$\stackrel{\downarrow}{=} \int dy\, f(y)^* \left[\int dx\, K(x,y;-t)^* \psi(x,0) \right]$

Since $K(x,y;-t)^* = K(y,x;t)$, this implies "b)" holds.

For "c)", let $R>0$, and define $\psi_R(x,0) = \mathbb{1}(|x| \leq R)\psi(x,0)$ and $\psi_R(t) := e^{-itH_0}\psi_R(0)$. As $\psi_R(x,0) \in L^1 \cap L^2$ and

$$\| \psi_R(t) - \psi(t) \| = \| e^{-itH_0}(\psi_R(0) - \psi(0)) \|$$
$$= \| \psi_R(0) - \psi(0) \| \to 0, \quad R \to \infty.$$

$\Rightarrow \exists$ sequence $R_n \to \infty$ s.t. for a.e. $x \in \mathbb{R}^d$

$$\psi(t,x) = \lim_{n\to\infty} \psi_{R_n}(t,x)$$
$$= \lim_{n\to\infty} \int dy\, K(x,y;t)\, \mathbb{1}(|y| \leq R_n)\, \psi(y,0).$$

Thus "c)" holds, as well. $\square$