5.13. Suppose the map $Q: I \to \mathcal{B}(\mathcal{H})$, $I \subset \mathbb{R}$ interval, is strongly continuous. $Q(t)$ is said to be strongly differentiable at t_0 for $\psi_0 \in \mathcal{H}$, if the following norm-lim. exists

$$\exists \lim_{\substack{t \to t_0 \\ t \in I}} \frac{1}{t-t_0}\left(Q(t)\psi_0 - Q(t_0)\psi_0\right) =: \frac{d}{dt} Q(t)\psi_0 \Big|_{t=t_0}$$

Defn. Suppose $(U(t))_{t\geq 0}$ is a strongly continuous unitary semi-group. Its infinitesimal generator is a map $A: D(A) \to \mathcal{H}$ defined using

$$D(A) := \{\psi \in \mathcal{H} \mid U(t) \text{ is strongly differentiable at } t=0 \text{ for } \psi\}$$

and for any $\psi \in D(A)$

$$A\psi := \lim_{\varepsilon \to 0^+} \frac{i}{\varepsilon}\left(U(\varepsilon)\psi - \psi\right) = i \frac{d}{dt} U(t)\psi \Big|_{t=0}$$

5.14. Theorem (Stone)

Suppose $(U(t))_{t\geq 0}$ is a strongly continuous unitary semigroup, and let A denote its infinitesimal generator, defined as above. Then A is a densely defined self-adjoint operator on $\mathcal{H}$ and $\forall t \geq 0$:

(Exp) $\quad U(t) = e^{-itA}$ (defined via spectral decomposition of A)

Denote $\psi(t) := U(t)\psi$ for $\psi \in \mathcal{H}$, $t \geq 0$.
Then a) $t \mapsto \psi(t)$ is norm-continuous.
b) If $\psi(0) \in D(A)$, then $\psi(t) \in D(A)$ $\forall t \geq 0$ and

$$i \frac{d}{dt}\psi(t) = A\psi(t) = U(t)A\psi(0).$$

c) $\forall \psi(0) \in \mathcal{H}$: $\quad \psi(t) = \lim_{\varepsilon \to 0^+} \exp\left(-it \frac{i}{\varepsilon}(U(\varepsilon)-1)\right)\psi(0)$.

Conversely, if A is self-adjoint, and $U(t) = e^{-itA}$, then $(U(t))_{t\geq 0}$ is a strongly continuous semigroup and A is its infinitesimal generator.

Proof: Functional calculus with spectral representations. (See the Appendix on page 55.). For complete proofs, see Rudin, Funct. Anal., Th. 13.35 and Th. 13.37 or Reed & Simon I, chapter VIII.4. or Teschl, chapter 3. or Hall, chapter 10. $\square$

Remarks: * The spectrum of a self-adjoint operator A, is a set $\sigma(A) \subset \mathbb{R}$. The spectral representation assigns to every Borel subset $\omega \subset \sigma(A)$ an orthogonal projection P_ω, so that for any $\phi, \psi \in \mathcal{H}$ the map $\mu_{\phi,\psi}: \omega \mapsto (\phi, P_\omega \psi)$ is a Borel measure, and $\forall \phi \in \mathcal{H}$, $\psi \in D(A)$

$$(\phi, A\psi) = \int_{\sigma(A)} \lambda \, \mu_{\phi,\psi}(d\lambda)$$

$$=: \int_{\sigma(A)} \lambda \, d(\phi, P_\lambda \psi)$$

The definition in (Exp) means

$\forall t \in \mathbb{R}, \phi, \psi \in \mathcal{H}$:

$$(\phi, e^{-itA}\psi) := \int_{\sigma(A)} e^{-it\lambda} \, d(\phi, P_\lambda \psi)$$

and the basic results of functional calculus show that then $D(e^{-itA}) = \mathcal{H}$ and e^{-itA} is unitary operator.

* If $\mathcal{H} = \mathbb{C}^N$, A is a self-adjoint matrix, with eigenvalues $\lambda_n \in \mathbb{R}$ and (orthonormal) collection of eigenvectors $e_n \in \mathbb{C}^N$ ...

$\longrightarrow$

... and the spectral definition means

$$(\phi, e^{-itA}\psi) := \sum_{n=1}^{N} e^{-it\lambda_n} (\phi, e_n)(e_n, \psi).$$

* If A is a bounded operator,

$$e^{-itA} = \sum_{n=0}^{\infty} \frac{1}{n!} (-itA)^n, \qquad (*)$$

but for unbounded operators, using the sum is usually not a good idea. For instance, if A is self-adj, usually $D(A^2) \subset D(A)$ is a proper subset, and the sum in $(*)$ makes sense only for so called analytic vectors; for ψ s.t.

$$\psi \in C^\infty(A) := \bigcap_{n=1}^{\infty} D(A^n) \quad \text{and}$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} \|A^n \psi\| t^n < \infty \quad \text{for some } t > 0.$$

* It is possible, that S is essentially self-adjoint, i.e., S is symmetric and $\bar{S}$ is self-adjoint, but although $C^\infty(\bar{S})$ is dense, $C^\infty(S)$ is just $\{0\}$.

* However, by c), e^{-itA} is a strong limit of

$$\sum_{n=0}^{\infty} \frac{1}{n!} (-itA_\varepsilon)^n \quad \text{where } A_\varepsilon = \frac{i}{\varepsilon}(U(\varepsilon) - I) \text{ is a bounded operator.}$$

5.15 Remark: Stone's theorem shows that the best we can do to understand the original Schrödinger equation $i\frac{d}{dt}\psi(t) = S\psi(t)$ on page 3 is to find a dense subspace of $\mathcal{H}$ for which the right hand side makes sense, and then look for self-adjoint extensions of S. As we will see later, even if S is symmetric, any of the following can happen:

1) $\bar{S}$ is the unique self-adjoint extension (S.A.E.)
2) There are (infinitely) many S.A.E.
3) There are no S.A.E.

* If 1) happens, we should just be happy.
* 2) means that we forgot to "put in all the physics" in the Schrödinger equation. Typical examples are boundary conditions.
* 3) means that the (physical) system is not closed, and we are either forced to "leak" or "inject probability". (roughly speaking)

5.16. Examples : Three standard ways of defining operators on $\mathcal{H} = L^2(\Omega)$ when $\Omega \subset \mathbb{R}^d$, open subset: multiplication, integral, and differential operators.

1) Multiplication operators (potentials)

Let $V: \Omega \to \mathbb{C}$ be Lebesgue measable. The corresponding multiplication operator M_V (also denoted $\hat{V}$ or simply V) is a mapping $D(M_V) \to \mathcal{H}$ defined by

$$(M_V\psi)(x) = V(x)\psi(x), \quad x \in \Omega$$

on

$$D(M_V) := \{\psi \in \mathcal{H} \mid V\psi \in L^2(\Omega)\}.$$

M_V has the following properties: (Ex. 5.2.)

a) M_V is a closed, densely defined operator.

b) $(M_V)^* = M_{V^*}$

c) M_V self-adjoint $\iff$ $V(x) \in \mathbb{R}$ a.e. $x \in \Omega$.

* Thus every $V: \Omega \to \mathbb{R}$, which is Lebesgue measurable, generates a strongly cont. USG. With $U_t = e^{-itM_V} = M_{(e^{-itV})}$.

2) Integral operators

Suppose $K : \Omega \times \Omega \to \mathbb{C}$ is Lebesgue measurable. For any $\psi \in \mathcal{H}$, let $\Xi_\psi \subset \Omega$ be the set of points in which $K(x,\cdot)\psi(\cdot) \in L^1(\Omega)$. For all ψ, for which Ξ_ψ^c has zero measure, we define

$$F_\psi(x) = \int_\Omega dy\, K(x,y)\psi(y), \quad x \in \Xi_\psi$$

and set (arbitrarily) $F_\psi(x) = 0$ for $x \notin \Xi_\psi$. Then the integral operator I_K corresponding to integral kernel K is defined on

$$D(I_K) := \{ \psi \in \mathcal{H} \mid \Xi_\psi^c \text{ has zero measure}, \int_\Omega |F_\psi|^2 < \infty \}$$

by $(I_K\psi)(x) = F_\psi(x)$.

* These are proper operators, but need not be densely defined or closed.

* Extremely nice special case:
  If $\int_\Omega dx \int_\Omega dy\, |K(x,y)|^2 < \infty$, then

  $I_K \in \mathcal{B}(\mathcal{H})$ and there is a sequence of finite-dimensional (rank) operators $F_K^{(n)}$

  $(\Leftrightarrow \dim(R(F_K^{(n)})) < \infty \quad \forall n)$

  s.t. $\lim_{n\to\infty} \| I_K - F_K^{(n)} \| = 0$.

  (Then I_K is a so called Hilbert-Schmidt operator.)

* If $K(x,y)^* = K(y,x)$ a.e. $x,y \in \Omega$ and $\exists C \geq 0$ s.t. $\int_\Omega dy\, |K(x,y)| \leq C$ a.e. $x \in \Omega$, then $I_K \in \mathcal{B}(\mathcal{H})$ and I_K is self-adjoint. (see Ex. 5.4.)

3) Differential operators

Let $(T\psi)(x) = \sum_{\alpha: |\alpha| \leq N} c_\alpha \partial^\alpha \psi(x)$

where $N > 0$, the sum goes over multi-indices α with degree up to N, and $c_\alpha \in \mathbb{C}$ $\forall |\alpha| \leq N$.

Then $T : D \to D$, at least for
$D = C_c^\infty(\Omega) = \{ f : \Omega \to \mathbb{C} \mid f$ smooth and supp f compact $\}$.

Since D is dense in $L^2(\Omega)$, T is clearly a densely defined operator on $\mathcal{H}$. However, it is not closed, so it cannot be self-adjoint.

* Standard example: $(T\psi)(x) = \nabla^2 \psi(x)$ (Laplacian).

* Closed extensions of differential operators is our next topic...

5.17. Definition: Arithmetics of unbounded operators:

Let A, B be operators.
The natural domains for $A+B$ and AB are

① $D(A+B) = D(A) \cap D(B)$ on which $(A+B)\psi = A\psi + B\psi$.

② $D(AB) = \{ \psi \in D(B) \mid B\psi \in D(A) \}$ on which $AB = A \circ B$.

* To be used with care: for instance, $H_0 = -\frac{1}{2}\nabla^2$ and $V = M_V$ can separately be fairly easily defined as self-adjoint operators. However, "$H_0 + V$" as defined above is not necessarily self-adjoint, even though it might be essentially self-adjoint, i.e., the Schrödinger operator is then really $H = \overline{H_0 + V}$.

Appendix: Spectral representations

The definition of $U(t)$ in our version of the Stone's theorem (5.14) uses the following general result:

5.17. Theorem: Suppose A is a self-adjoint operator on a Hilbert space $\mathcal{H}$. Then there is a unique projection value measure E, on the Borel subsets of $\sigma(A)$, the spectrum of A, such that

$$(\phi, A\psi) = \int_{\sigma(A)} E_{\phi,\psi}(d\lambda)\, \lambda \qquad \forall \psi \in D(A),\ \phi \in \mathcal{H}.$$

Moreover, then $\sigma(A)$ is a non-empty subset of $\mathbb{R}$.

Proof: Rudin, F.A., Theorem 13.30, or Teschl, Theorem 3.7. $\square$

* This generalization of diagonalization of Hermitean matrices is a central result of modern functional analysis. Its real mathematical content is hidden in the definitions:

5.18. Definition: Suppose T is an operator on a Hilbert space $\mathcal{H}$. Its spectrum $\sigma(T)$ is the collection of those $\lambda \in \mathbb{C}$ for which $\lambda 1 - T$ is not invertible in $\mathcal{B}(\mathcal{H})$.

* In other words, $\lambda \notin \sigma(T)$ iff $R(\lambda 1 - T) = \mathcal{H}$, $\lambda 1 - T$ is one-to-one, and $\exists S \in \mathcal{B}(\mathcal{H})$ such that $S = (\lambda 1 - T)^{-1}$.

* If T is unbounded, it is possible that $\sigma(T) = \emptyset$. However, $\sigma(T)$ is always a closed subset of $\mathbb{C}$.

* The complement of $\sigma(T)$ is called the resolvent set, denoted by $\rho(T)$. To every $\lambda \in \rho(T)$ we know that $(\lambda - T)^{-1} \in \mathcal{B}(\mathcal{H})$. The corresponding map $\rho(T) \to \mathcal{B}(\mathcal{H})$ is called the resolvent of T.

* If T is closed, it suffices to check that $\lambda 1 - T$ is bijective (the first two items above). (By the closed graph theorem, then the inverse $(\lambda 1 - T)^{-1}$ is continuous, i.e., belongs to $\mathcal{B}(\mathcal{H})$.)

* If $\lambda 1 - T$ is not one-to-one, then there is $\psi \in \mathcal{H}$, $\psi \neq 0$, s.t. $T\psi = \lambda\psi$. Obviously, then $\lambda \in \sigma(T)$, and we say that λ is an eigenvalue of T and ψ is an eigenvector of T corresponding to λ.

* A subspace $M \subset \mathcal{H}$ is called an invariant subspace of T, if M is closed and $TM \subset M$. Clearly, if λ is an eigenvalue, then $\ker(\lambda 1 - T)$ is an invariant subspace, called the eigenspace of T corresponding to λ.

* If T is normal (e.g. self-adjoint), its spectrum typically contains $\lambda \in \sigma(T)$ which are not eigenvalues. It is, however, an approximate eigenvalue: To every $\varepsilon > 0$, there is $\psi \in D(T)$ such that

$$\|\psi\| = 1 \quad \text{and} \quad \|T\psi - \lambda\psi\| \leq \varepsilon .$$

(Rudin, F.A., Thrm 13.27. See also Hall, Proposition 10.8.)

* We still need to define what is a projection valued measure, also called a resolution of the identity (e.g., Rudin). The terminology is not completely fixed (c.f. Teschl), so it is a good idea to always check them.
* We use the definition relevant to the general spectral decompositions, as in Rudin.
* As discussed during the lectures, there is some redundancy in the definition of PVMs below. For instance, item "c)" can be dropped. For more details, see Teschl, p. 88-89.

5.19. Definition: Suppose $\mathcal{M}$ is a σ-algebra in a set X, and $\mathcal{H}$ is a Hilbert space. A mapping $E: \mathcal{M} \to \mathcal{B}(\mathcal{H})$ is a projection valued measure (PVM) if it satisfies

a) $E(\emptyset) = 0$, $E(X) = 1 = id_{\mathcal{H}}$

b) $E(\omega)$ is a self-adjoint projection $\forall \omega \in \mathcal{M}$

c) $E(\omega \cap \omega') = E(\omega)E(\omega')$ $\quad \forall \omega, \omega' \in \mathcal{M}$

d) If $\omega, \omega' \in \mathcal{M}$ and $\omega \cap \omega' = \emptyset$, then $E(\omega \cup \omega') = E(\omega) + E(\omega')$.

e) $\forall \psi, \varphi \in \mathcal{H}$ the map $E_{\varphi,\psi}: \mathcal{M} \to \mathbb{C}$ defined by

$$E_{\varphi,\psi}(\omega) := (\varphi, E(\omega)\psi)$$

is a complex measure on $\mathcal{M}$.

If X = locally compact Hausdorff space and $\mathcal{M}$ = Borel σ-algebra on X, we say that E is a regular resolution of the identity, if every $E_{\varphi,\psi}$ is a regular Borel measure (see 2.15 and 6.19 in [R(A)])

8.20. Proposition: Suppose $\mathcal{H}$ is a Hilbert space, $\mathcal{M}$ is a σ-algebra in X, and $E: \mathcal{M} \to \mathcal{B}(\mathcal{H})$ is a PVM. Define $E_{\varphi,\psi}$ by e) above for $\varphi, \psi \in \mathcal{H}$. Consider an arbitrary $\psi_0 \in \mathcal{H}$. Then

a) E_{ψ_0,ψ_0} is a bounded positive measure on $\mathcal{M}$ whose total variation is $\|\psi_0\|^2$, and $E_{\psi_0,\psi_0}(\omega) = \|E(\omega)\psi_0\|^2 \; \forall \omega \in \mathcal{M}$.

b) $E(\omega)$ and $E(\omega')$ commute $\forall \omega, \omega' \in \mathcal{M}$

c) If $\omega, \omega' \in \mathcal{M}$ are disjoint, then $R(E(\omega)) \perp R(E(\omega'))$.

d) E is finitely additive.

e) the map $\omega \mapsto E(\omega)\psi_0$ is a countably additive $\mathcal{H}$-valued measure on $\mathcal{M}$.

f) If $\omega_n \in \mathcal{M}$, $n \in \mathbb{N}_+$, all satisfy $E(\omega_n) = 0$, then $E(\bigcup_{n\in\mathbb{N}_+} \omega_n) = 0$.

Proof: "a)" If $\omega \in \mathcal{M} \Rightarrow E(\omega)^* = E(\omega) = E(\omega)^2 \Rightarrow$

$$E_{\psi_0,\psi_0}(\omega) = (\psi_0, E(\omega)\psi_0) = (\psi_0, E(\omega)^* E(\omega)\psi_0)$$
$$= (E(\omega)\psi_0, E(\omega)\psi_0) = \|E(\omega)\psi_0\|^2 \geq 0$$

$\Rightarrow E_{\psi_0,\psi_0}(X) = \|\psi_0\|^2 < \infty$. Thus E_{ψ_0,ψ_0} is a bounded positive measure with $\|E_{\psi_0,\psi_0}\| = \|\psi_0\|^2$. ∎

"b)" If $\omega, \omega' \in \mathcal{M} \overset{c)}{\Rightarrow} E(\omega)E(\omega') = E(\omega \cap \omega') = E(\omega' \cap \omega) = E(\omega')E(\omega)$.

"c)" If $\omega \cap \omega' = \emptyset \Rightarrow 0 \overset{a)}{=} E(\emptyset) = E(\omega \cap \omega') \overset{c)}{=} E(\omega)E(\omega') =$

$\Rightarrow \forall \varphi, \psi \in \mathcal{H}: \; 0 = (\varphi, E(\omega)E(\omega')\psi) = (E(\omega)^*\varphi, E(\omega')\psi)$
$= (E(\omega)\varphi, E(\omega')\psi) \Rightarrow R(E(\omega)) \perp R(E(\omega'))$.

"d) & e)" If $\omega_n \in \mathcal{M}$, $n \in \mathbb{N}_+$, are disjoint, then $\forall N \in \mathbb{N}_+$

$$\omega_{N+1} \cap \left(\bigcup_{n=1}^{N} \omega_n\right) = \bigcup_{n=1}^{N} (\omega_{N+1} \cap \omega_n) = \emptyset \Rightarrow$$

$E\left(\bigcup_{n=1}^{N+1} \omega_n\right) \overset{d)}{=} E(\omega_{N+1}) + E\left(\bigcup_{n=1}^{N} E_n\right)$. Thus by induction

$\Rightarrow E\left(\bigcup_{n=1}^{N} \omega_n\right) = \sum_{n=1}^{N} E(\omega_n) \;\; \forall N \in \mathbb{N}_+$. Proves "d)".

(typically not countably additive, since by Ex. 3.4. here either $E(\omega_n) = 0$ or $\|E(\omega_n)\| = 1 \Rightarrow$ the sum is not norm-Cauchy unless $E(\omega_n) = 0 \; \forall n \geq n_0$.)

To prove "e)" it suffices to show that now

$E\left(\bigcup_{n\in\mathbb{N}_+} \omega_n\right)\psi_0 = \sum_{n=1}^{\infty} E(\omega_n)\psi_0$ as an $\mathcal{H}$-convergent sum.

For this, denote $\varphi_n := E(\omega_n)\psi_0 \in R(E(\omega_n))$. By "c)" $\Rightarrow \varphi_n \perp \varphi_m \quad \forall n \neq m$. $\Rightarrow$ if $I \subset \mathbb{N}_+$ and $|I| < \infty$ then

$$\| \sum_{n \in I} \varphi_n \|^2 = \sum_{n', n \in I} (\varphi_{n'}, \varphi_n) = \sum_{n \in I} \|\varphi_n\|^2$$

$$= \sum_{n \in I} (E(\omega_n)\psi_0, E(\omega_n)\psi_0) = \sum_{n \in I} (\psi_0, E(\omega_n)\psi_0)$$

$$= \sum_{n \in I} E_{\psi_0,\psi_0}(\omega_n) = E_{\psi_0,\psi_0}(\bigcup_{n \in I} \omega_n) \overset{\text{"a)"}}{\leq} \|\psi_0\|^2$$

$\Rightarrow \sum_{n \in \mathbb{N}_+} \|\varphi_n\|^2 \leq \|\psi_0\|^2 \Rightarrow \forall \varepsilon > 0 \; \exists n_0$ s.t. $\forall m \geq 0$

$$\| \sum_{n=n_0}^{n_0+m} \varphi_n \|^2 = \sum_{n=n_0}^{n_0+m} \|\varphi_n\|^2 \leq \varepsilon^2 \Rightarrow \left(\sum_{n=1}^{N} \varphi_n\right)_{N \in \mathbb{N}_+} \text{ is}$$

Cauchy in $\mathcal{H}$ $\Rightarrow \exists \tilde{\psi} = \lim_{N \to \infty} \sum_{n=1}^{N} \varphi_n$. $\Rightarrow \forall \psi \in \mathcal{H}$:

$$(\psi, \tilde{\psi}) = \lim_{N \to \infty} \sum_{n=1}^{N} (\psi, E(\omega_n)\psi_0) = \sum_{n=1}^{\infty} E_{\psi,\psi_0}(\omega_n)$$

$$= E_{\psi,\psi_0}\left(\bigcup_{n=1}^{\infty} \omega_n\right) = (\psi, E(\cup \omega_n)\psi_0)$$

$$\Rightarrow \tilde{\psi} = \sum_{n=1}^{\infty} E(\omega_n)\psi_0 = E\left(\bigcup_{n=1}^{\infty} \omega_n\right)\psi_0 \quad \blacksquare$$

"f)" Let $\omega := \bigcup_{n=1}^{\infty} \omega_n \in M$. Then $\forall \psi \in \mathcal{H}$, $E_{\psi,\psi}$ is a measure

$$\overset{\text{"a)"}}{\Rightarrow} \|E(\omega)\psi\|^2 = E_{\psi,\psi}(\omega) = \int E_{\psi,\psi}(d\lambda)\, \mathbb{1}(\lambda \in \omega)$$

$$\overset{\text{DCT}}{=} \lim_{N \to \infty} \int E_{\psi,\psi}(d\lambda)\, \underbrace{\mathbb{1}(\lambda \in \bigcup_{n=1}^{N} \omega_n)}_{\leq \sum_{n=1}^{N} \mathbb{1}(\lambda \in \omega_n)} \leq \sum_{n=1}^{\infty} \int E_{\psi,\psi}(d\lambda)\, \mathbb{1}(\lambda \in \omega_n)$$

$\Rightarrow \|E(\omega)\psi\|^2 \leq \sum_{n=1}^{\infty} E_{\psi,\psi}(\omega_n)$, where $E_{\psi,\psi}(\omega_n)$
$= (\psi, E(\omega_n)\psi) = 0$ as $E(\omega_n) = 0$.
$\Rightarrow 0 \leq \|E(\omega)\psi\|^2 \leq 0 \Rightarrow E(\omega)\psi = 0$. $\therefore E(\omega) = 0 \quad \square$

* The name "projection valued measure" is used since each $E(\omega)$ is a projection and, by "e)" above, each $\omega \mapsto E(\omega)\psi$ is an $\mathcal{H}$-valued measure.

* the above formulation of the spectral decomposition theorem is the one directly relevant to the Stone's theorem. There are many useful generalizations; three are listed below:

5.21. Definition: Suppose T is an operator on a Hilbert space $\mathcal{H}$. T is called normal, if it is densely defined, closed, and satisfies $T^*T = TT^*$.

* This obviously generalizes the definition of bounded normal operators, given in 3.4.

5.22. Theorem: Suppose T is a normal operator on $\mathcal{H}$. Then there is E, a unique PVM on $\sigma(T)$, called the spectral decomposition of T, which satisfies

$$(\phi, T\psi) = \int_{\sigma(T)} E_{\phi,\psi}(d\lambda)\, \lambda \quad \forall \psi \in D(T), \phi \in \mathcal{H}.$$

Moreover, if $S \in \mathcal{B}(\mathcal{H})$ commutes with T

($\Leftrightarrow$ $ST \subset TS$), then $[S, E(\omega)] = 0$ for all Borel sets $\omega \subset \sigma(T)$.

Proof: Rudin, F.A., Theorem 13.33. $\square$

* It is possible to "diagonalize" bounded normal operators simultaneously, if the operators commute. This is explained in the following theorem. In physics, one can think of the set "K" in the theorem as a collection of "quantum numbers" related to the family of normal operators (observables).

* There is also a version of the theorem for normal <u>unbounded</u> operators. This, however, will <u>require</u> more careful definition of what "commutation" of two unbounded operators means: it is possible that $D(AB) = \{0\}$ even if A and B are densely defined.

The definition relevant to the spectral decomposition is to check that some sufficiently large family of bounded operators generated by the normal operators A, B commute. For instance, if A, B are <u>self-adjoint</u> it suffices to check that

either a) $\left[\frac{1}{\lambda - A}, \frac{1}{\lambda' - B}\right] = 0 \quad \forall \lambda, \lambda' \in \mathbb{C}$ for which $\operatorname{Im}\lambda, \operatorname{Im}\lambda' \neq 0$.
(resolvents commute)

or b) $\left[e^{-itA}, e^{-it'B}\right] = 0 \quad \forall t, t' \in \mathbb{R}$.

i.e. that the semigroups generated by A and B commute.

Then it follows that $[f(A), g(B)] = 0$ for any bounded Borel functions f, g. (Proof: Reed & Simon I, Theorem VIII.13.)

It does <u>not</u> suffice to check that $AB\psi = BA\psi$ for some dense collection of ψ (RS I, p. 273).

* For finitely many commuting bounded normal we have:

5.23. Theorem: Suppose $T_i \in \mathcal{B}(\mathcal{H})$, $i = 1, \dots, N$, $N < \infty$, are all <u>normal</u> operators which commute pairwise. Then $\exists$ nonempty compact set $K \subset \mathbb{C}^N$ and a regular PVM E on the Borel σ-algebra of K such that

$$(\phi, T_i \psi) = \int_K E_{\phi,\psi}(d\lambda)\, \lambda_i \quad \forall i = 1, \dots, N, \ \phi, \psi \in \mathcal{H}.$$

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...

In addition, then $\lambda \in K \Rightarrow \lambda_i \in \sigma(T_i)\ \forall i=1,\dots,N$ and if S commutes with every T_i, then S commutes with every $E(\omega)$.

Proof: Application of the general Gelfand-theory, Rudin, F.A., Theorem 12.22. □
(Justified by its Theorem 12.16.)

* The following result shows how to use PVMs (such as spectral decompositions) to generate new operators. This is also how "e^{-itA}", A self-adjoint, is defined in the Stone's theorem.

5.24. Theorem: Suppose E is a PVM from the σ-algebra $\mathcal{M}$ on the set X to projection operators on the Hilbert space $\mathcal{H}$. Then to every measurable $f: X \to \mathbb{C}$ there is a unique normal operator $\mathcal{O}(f)$ on $\mathcal{H}$ with the domain

$$D(\mathcal{O}(f)) := \{ \psi \in \mathcal{H} \mid \int_X E_{\psi,\psi}(d\lambda)\, |f(\lambda)|^2 < \infty \}$$

and satisfying

$$(\phi, \mathcal{O}(f)\psi) = \int_X E_{\phi,\psi}(d\lambda)\, f(\lambda) \quad \forall \psi \in D(\mathcal{O}(f)), \phi \in \mathcal{H}.$$

In addition, the following properties hold for any f, g which are measurable:

a) $\| \mathcal{O}(f)\psi \|^2 = \int_X E_{\psi,\psi}(d\lambda)\, |f(\lambda)|^2 \quad \forall \psi \in D(\mathcal{O}(f))$

b) $\mathcal{O}(f^*) = \mathcal{O}(f)^*$

c) $\mathcal{O}(|f|^2) = \mathcal{O}(f)^* \mathcal{O}(f) = \mathcal{O}(f)\mathcal{O}(f)^*$

d) $\mathcal{O}(f)\mathcal{O}(g) \subset \mathcal{O}(fg)$ where $D(\mathcal{O}(f)\mathcal{O}(g)) = D(\mathcal{O}(fg)) \cap D(\mathcal{O}(g))$

Proof: Rudin, F.A., Theorem 13.24. □

* If T is normal and E its spectral decomposition, then it is customary to write "$f(T)$" instead of "$\mathcal{O}(f)$" above. This is sometimes referred to as "symbolic calculus".
* If f is a bounded measurable function, $D(\mathcal{O}(f)) = \mathcal{H}$, and since f is closed, we then have $\mathcal{O}(f) \in \mathcal{B}(\mathcal{H})$. Therefore, by "d)" above, if f, g are both bounded, then $\mathcal{O}(f)\mathcal{O}(g) = \mathcal{O}(fg)$ and thus $[\mathcal{O}(f), \mathcal{O}(g)] = 0$.

* Suppose that $\mathcal{H}$ and $\mathcal{H}'$ are Hilbert spaces and $\mathcal{U} : \mathcal{H} \to \mathcal{H}'$ is a unitary map. If A is an operator on $\mathcal{H}$ and A' an operator on $\mathcal{H}'$, we say that they are unitarily equivalent if

$$\mathcal{U}A = A'\mathcal{U} \quad \text{and} \quad UD(A) = D(A').$$

Any Hermitian matrix can be diagonalized by a unitary matrix. The following result is the closest equivalent for Hilbert spaces whose dimension is countable.

5.25. Theorem: Suppose A is a self-adjoint operator on a separable Hilbert space $\mathcal{H}$. Then we can decompose $\mathcal{H}$ into orthogonal subspaces $\mathcal{H}_n$, $n \in I$, where I is finite or countably infinite, such that to each $n \in I$ there is a positive Borel measure μ_n on $\mathbb{R}$ and a unitary map $U_n : \mathcal{H}_n \to L^2(\mu_n)$ which turns A into a multiplication operator: for every $\psi \in D(A) \cap \mathcal{H}_n$ we have $(U_n A\psi)(x) = x\,(U_n\psi)(x)$.

In addition, it is possible to choose μ_n, U_n so that $\mathcal{U} := \bigoplus_n U_n$ is a unitary map from $\mathcal{H} = \bigoplus_n \mathcal{H}_n$ to $\bigoplus_n L^2(\mathbb{R}, \mu_n)$ and $f(A)$ on $\mathcal{H}$ is unitarily equivalent to the operator $\bigoplus_n f_n$ where f_n denotes the multiplication operator f on $L^2(\mathbb{R}, \mu_n)$.

Proof: Teschl, Lemma 3.4. or R&S I, Theorem VII.3 □

5.26 Theorem: Suppose E is a PVM from the σ-algebra $\mathcal{M}$ on the set $\mathfrak{X}$ to projection operators on the Hilbert space $\mathcal{H}$. If S is a bounded operator which commutes with every projection $E(\omega)$, then S commutes with every $\mathcal{O}(f)$ where $f: \mathfrak{X} \to \mathbb{C}$ is measurable and bounded.

Proof: Spectral theory course, or Rudin, F.A., theorem 12.21. $\square$

* In summary, the above theorem states that

[If $S \in \mathcal{B}(\mathcal{H})$ and $[S, E(\omega)] = 0$ $\forall \omega \in \mathcal{M}$, then $[S, \mathcal{O}(f)] = 0$ $\forall$ measurable $f: \mathfrak{X} \to \mathbb{C}$ for which $\sup_{x \in \mathfrak{X}} |f(x)| < \infty$.]

* Combining theorem 5.26. with Stone's theorem and Theorem 5.22. yields the following general result about quantum dynamics:

[If a bounded operator commutes with the inf. generator of the dynamics, then it commutes also with the time-evolution.]

More precisely, we obtain:

5.27. Corollary: Let $(U_t)_{t \geq 0}$ be a strongly continuous unitary semigroup, and let H denote its infinitesimal generator, so that $U_t = e^{-itH}$.

If $S \in \mathcal{B}(\mathcal{H})$ satisfies $SH \subset HS$, then $SU_t = U_t S$ for all $t \in \mathbb{R}$.

Proof: By 5.14, H is self-adjoint $\Rightarrow$ H is normal and $\sigma(H) \subset \mathbb{R}$. By 5.22., $\exists$ P.V.M. E on $\sigma(H)$ and $SH \subset HS$ $\Rightarrow$ $[S, E(\omega)] = 0$ $\forall \omega \in \mathcal{M}$. Since $U_t = \mathcal{O}(f_t)$ for $f_t(\lambda) := e^{-it\lambda}$, $\lambda \in \sigma(H)$ $\Rightarrow$ $|f_t(\lambda)| = 1$ $\forall t, \lambda$, by 5.26. we have $[S, U_t] = 0$. $\square$