2.20. Spin

Definition: Spin-s particles

Assume $s \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\}$.

* The wave-function of a spin-s particle is a $(2s+1)$-component function on $L^2(d^3x)$:

$$\psi \in L^2(\mathbb{R}^3, \mathbb{C}^{2s+1}).$$

Then $\|\psi\|^2 = \int_{\mathbb{R}^3} dx \sum_{\sigma=1}^{2s+1} |\psi_\sigma(x)|^2$.

* Most of the time one uses a labeling $\{-s, -s+1, \dots, s\} =: I$ i.e.:

  a) for spin-$\frac{1}{2}$ particles: $\{-\frac{1}{2}, \frac{1}{2}\}$ or $\{\downarrow, \uparrow\}$ or $\{-, +\}$

  b) for spin-1 partices: $\{-1, 0, 1\}$

  ...

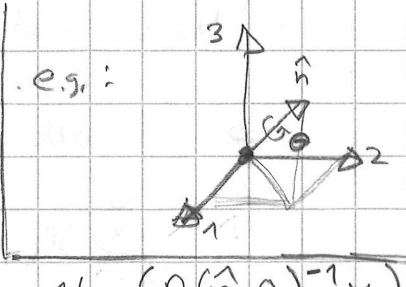
* $L^2(\mathbb{R}^3, \mathbb{C}^I) \cong \bigoplus_{\sigma \in I} L^2(\mathbb{R}^3) \cong L^2(\mathbb{R}^3) \otimes \mathbb{C}^I$

  (See Ex. 3.5.a)

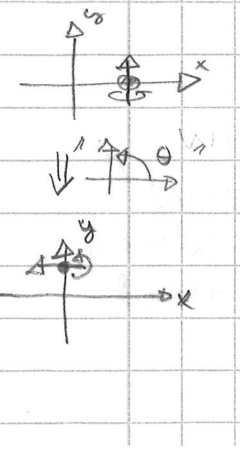
* Spin-components transform nontrivially under rotations of $\mathbb{R}^3$ (unless $s=0$):

Example: A rotation of $\mathbb{R}^3$ of "angle" θ around a direction $\hat{n} \in \mathbb{R}^3$ ($|\hat{n}|=1$) acts on a wave-function $\psi \in L^2(\mathbb{R}^3, \mathbb{C}^2)$ of a spin-$\frac{1}{2}$ particle by the rule $\psi \mapsto \tilde{\psi}$

e.g.:



Original motivation:



with $\tilde{\psi}_\sigma(x) := \sum_{\sigma' \in \{\pm\frac{1}{2}\}} D(\hat{n}, \theta)_{\sigma\sigma'} \psi_{\sigma'}(R(\hat{n}, \theta)^{-1} x)$, $x \in \mathbb{R}^3$, $\sigma \in \{\pm\frac{1}{2}\}$

where $D(\hat{n}, \theta) := e^{-i\frac{\theta}{2} \sum_{j=1}^{3} \underbrace{\hat{n}_j}_{\in \mathbb{R}} \underbrace{\hat{\sigma}_j}_{\in \mathbb{C}^{2\times 2}}}$ and

$\hat{\sigma}_j$, $j=1,2,3$, are called the Pauli matrices,

defined by $\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$R(\hat{n},\theta)$ denotes the corresponding rotation map which transforms vectors $v \mapsto \tilde{v}$

with $\tilde{v}_i := \sum_{j=1}^{3} R(\hat{n},\theta)_{ij} v_j$ where

$R(\hat{n},\theta) \in \mathbb{R}^{3\times 3}$ is the orthogonal matrix

$$R(\hat{n},\theta) := e^{\theta \sum_{j=1}^{3} \hat{n}_j \hat{A}_j}$$

and $\hat{A}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$, $\hat{A}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, $\hat{A}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

(note: "$\hat{A}_i$" corresponds to "$-\frac{i}{2}\hat{\sigma}_i$")

* Exercise: show that $R(\hat{n},\theta+2\pi) = R(\hat{n},\theta)$ but $D(\hat{n},\theta+2\pi) = -D(\hat{n},\theta)$.
* Exercise: Show that the map $\psi \mapsto \tilde{\psi}$ is unitary.

2.21. Multiparticle states

Consider N particles, where particle k has spin s_k, and thus its wave function belongs to $\mathcal{H}_k := L^2(\mathbb{R}^3, \mathbb{C}^{2s_k+1})$ $(k=1,\dots,N)$.

Def. The total wave function of the system of N particles is an element in $\bigotimes_{k=1}^{N} \mathcal{H}_k$.

* Note that $\bigotimes_{k=1}^{N} \mathcal{H}_k \cong L^2(\mathbb{R}^{3N}, \mathbb{C}^{\prod_{k=1}^{N}(2s_k+1)})$, $\cong \prod_{k=1}^{N} \mathbb{C}^{2s_k+1}$

$\Rightarrow$ if all particles have the same spin:

$$\bigotimes_{k=1}^{N} \mathcal{H}_k \cong L^2(\mathbb{R}^{3N}, \mathbb{C}^{(2s+1)^N}).$$

However, not all states are always "physical". (For instance, for bosons and fermions which have extra symmetry requirements. More about them later...)

3. Bounded operators: $\mathcal{B}(\mathcal{H})$

* As in the general case in 2.2., we define for $V = \mathcal{H}$ the space

$$\mathcal{B}(\mathcal{H}) = \{ T : \mathcal{H} \to \mathcal{H} \mid T \text{ linear and } \|T\| < \infty \}$$

with $\|T\| = \sup \{ \|T\psi\| \mid \psi \in \mathcal{H}, \|\psi\| = 1 \}$.

* An <u>operator</u> on $\mathcal{H}$ is a linear mapping

$A : D \to \mathcal{H}$, with $D \subset \mathcal{H}$ subspace.

$D = D(A) =$ <u>domain</u> of A.
$R(A) = \{ A\psi \mid \psi \in D \} =$ <u>range</u> of A.
$\operatorname{Ker}(A) = \{ \psi \in D \mid A\psi = 0 \} =$ <u>null space</u> or <u>kernel</u> of A

* $\mathcal{B}(\mathcal{H})$ is also called the set of <u>bounded operators</u>. Note that $T \in \mathcal{B}(\mathcal{H})$ implies $D(T) = \mathcal{H}$. (However, by Ex. 2.4., if $D(T)$ is dense, it has a unique extension to $\bar{T} \in \mathcal{B}(\mathcal{H})$.)

* $\mathcal{B}(\mathcal{H}) = \{$ the set of contiuous linear transformations of $\mathcal{H}\}$ (by Ex. 1.1.)

* $\mathcal{B}(\mathcal{H})$ is a Banach space, since $\mathcal{H}$ is complete.

* The following yields an important classification of bounded linear and sesquilinear functionals on $\mathcal{H}$:

<u>3.1. Thm:</u> a) Suppose $\Lambda : \mathcal{H} \to \mathbb{C}$ is <u>linear</u> and <u>bounded</u>. Then $\exists!\ \psi_0 \in \mathcal{H}$ s.t.

$$\Lambda\psi = (\psi_0, \psi) \quad \forall \psi \in \mathcal{H}.$$

b) Suppose $\Gamma : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is <u>bounded</u> and <u>sesquilinear</u>; that is, assume that

$\longrightarrow$

(i) $\Gamma(\phi, \alpha\psi_1 + \beta\psi_2) = \alpha\Gamma(\phi,\psi_1) + \beta\Gamma(\phi,\psi_2)$,
$\Gamma(\alpha\phi_1 + \beta\phi_2, \psi) = \alpha^*\Gamma(\phi_1,\psi) + \beta^*\Gamma(\phi_2,\psi)$
$\forall \phi, \phi_1, \phi_2, \psi, \psi_1, \psi_2 \in \mathcal{H}$, $\alpha, \beta \in \mathbb{C}$.

(ii) $\exists C \geq 0$ s.t. $|\Gamma(\phi,\psi)| \leq C\|\phi\|\,\|\psi\|$ $\forall \phi,\psi \in \mathcal{H}$.

Then $\exists!\, T \in \mathcal{B}(\mathcal{H})$ s.t. $\forall \phi, \psi \in \mathcal{H}$

$$\Gamma(\phi,\psi) = (\phi, T\psi)$$

and $\|T\| = \sup\{|\Gamma(\phi,\psi)| \mid \|\phi\| = 1 = \|\psi\|\} \leq C$.

pf: a) Boundedness of Λ means that (compare to 2.2.)
$\|\Lambda\| := \sup\{|\Lambda\psi| \mid \psi \in \mathcal{H}, \|\psi\| = 1\} < \infty$.
Let us start with uniqueness: If $\psi_0, \psi_0' \in \mathcal{H}$ s.t. $(\psi_0, \psi) = (\psi_0', \psi)$ $\forall \psi$ $\Rightarrow$
$(\psi_0 - \psi_0', \psi) = 0$ $\forall \psi$ $\Rightarrow$
$0 = (\psi_0 - \psi_0', \psi_0 - \psi_0') = \|\psi_0 - \psi_0'\|^2$
$\Rightarrow \psi_0' = \psi_0$. Thus ψ_0 is unique.
If $\Lambda = 0 \Rightarrow \Lambda\psi = 0 = (0,\psi)$ $\forall \psi \Rightarrow \psi_0 = 0$ is ok. If $\Lambda \neq 0$, let $M = \ker(\Lambda)$ $:= \{\psi \in \mathcal{H} \mid \Lambda\psi = 0\}$. Since Λ is bounded $\Rightarrow \Lambda$ continuous $\Rightarrow M = \Lambda^{\leftarrow}(\{0\})$ is closed. It is also obviously a subspace. By Thm. 2.11. $\mathcal{H} = M \oplus M^\perp$. Now $M^\perp \neq \{0\}$ since else $M = \mathcal{H}$ which would mean $\Lambda = 0$. Thus $\exists \phi \in M^\perp$, $\phi \neq 0$.
However, then for any $\psi \in \mathcal{H}$
$\Lambda((\Lambda\psi)\phi - (\Lambda\phi)\psi) = (\Lambda\psi)(\Lambda\phi) - (\Lambda\phi)(\Lambda\psi) = 0$
$\Rightarrow (\Lambda\psi)\phi - (\Lambda\phi)\psi \in M$. Then by $\phi \in M^\perp$
$\Rightarrow (\Lambda\psi)\underbrace{(\phi,\phi)}_{\neq 0} - (\Lambda\phi)(\phi,\psi) = 0$

$$\Rightarrow \Lambda\psi = \frac{\Lambda\phi}{\|\phi\|^2}(\phi,\psi) = (\psi_0,\psi)$$

for $\psi_0 = \dfrac{(\Lambda\phi)^*}{\|\phi\|^2}\phi$. This proves a). ■

b) Is a corollary of a): For any $\psi \in \mathcal{H}$ by (i),(ii) the map $\Lambda_\psi : \phi \mapsto \Gamma(\phi,\psi)^*$ is linear, and $|\Lambda_\psi \phi| \leq C\|\psi\|$ if $\|\phi\| = 1$.

$\longrightarrow$

Thus by a), $\exists ! \, \psi_0 \in \mathcal{H}$ s.t. $\Lambda_\psi \phi = (\psi_0, \phi) \; \forall \phi \in \mathcal{H}$.
We denote the map $\psi \mapsto \psi_0$ by T, when

$$\forall \phi, \psi \in \mathcal{H} : \; (T\psi, \phi) = \Lambda_\psi \phi = \Gamma(\phi, \psi)^*$$
$$\Rightarrow \; \Gamma(\phi, \psi) = (\phi, T\psi).$$

By linearity of $\Gamma(\phi, \cdot) \Rightarrow$

$$\forall \phi \in \mathcal{H} : \; (\phi, T(\alpha\psi_1 + \beta\psi_2)) = \Gamma(\phi, \alpha\psi_1 + \beta\psi_2)$$
$$= \alpha(\phi, T\psi_1) + \beta(\phi, T\psi_2)$$
$$\Rightarrow T(\alpha\psi_1 + \beta\psi_2) = \alpha T\psi_1 + \beta T\psi_2.$$

thus T is linear. Also

$$\|T\psi\|^2 = (T\psi, T\psi) = \Gamma(T\psi, \psi) \leq C \|T\psi\| \|\psi\|.$$
$$\Rightarrow \text{ if } \|\psi\| = 1, \quad \|T\psi\| \leq C \Rightarrow \|T\| \leq C < \infty.$$

$\Rightarrow T \in \mathcal{B}(\mathcal{H})$ and, as for any $\phi \in \mathcal{H}$

$$\|\phi\| = \sup \{ |(\phi', \phi)| \mid \phi' \in \mathcal{H}, \|\phi'\| = 1 \}$$

(Proof: Cauchy-Schwarz),

we also have

$$\|T\| = \sup \{ \|T\psi\| \mid \|\psi\| = 1 \}$$
$$= \sup \{ |(\phi, T\psi)| \mid \|\psi\| = 1 = \|\phi\| \}$$
$$= \sup \{ |\Gamma(\phi, \psi)| \mid \|\psi\| = 1 = \|\phi\| \} \leq C.$$

For uniqueness: If $T' \in \mathcal{B}(\mathcal{H})$ is s.t.

$$\Gamma(\phi, \psi) = (\phi, T'\psi) \; \forall \phi, \psi \Rightarrow$$
$$0 = (\phi, T'\psi - T\psi) \; \forall \phi, \psi \Rightarrow T'\psi = T\psi \; \forall \psi$$
$$\Rightarrow T' = T. \; \square$$

* "a)" is called "Riesz lemma" or "Riesz reprentation theorem" (e.g. Wikipedia). It implies that the dual of $\mathcal{H}$ is $\mathcal{H}$ itself.

3.2. Adjoint of a bounded operator

If $T \in \mathcal{B}(\mathcal{H})$, for all $\phi, \psi \in \mathcal{H}$

$$|(\phi, T\psi)| \overset{C\text{-}S.}{\leq} \|\phi\| \|T\psi\| \leq \|T\| \|\phi\| \|\psi\|$$

since $\|T\psi\| \leq \|T\| \|\psi\|$. (If $\psi = 0 \Rightarrow T\psi = 0$
$\Rightarrow \|T\psi\| = 0 = \|T\| \|\psi\|$. Else $\|T\psi\| = \|T \underbrace{\tfrac{\psi}{\|\psi\|}}_{\|\,\| = 1}\| \|\psi\|$
$\leq \|T\| \|\psi\|$, by definition of $\|T\|$.)

therefore, $\Gamma(\phi, \psi) = (T\phi, \psi) = (\psi, T\phi)^*$ satisfies the assumptions of Th. 3.1, b).

$\Rightarrow \exists!\, T^* \in \mathcal{B}(\mathcal{H})$ s.t. $(T\phi, \psi) = (\phi, T^*\psi)$ $\forall \phi, \psi$.

Also it follows that $\|T^*\| = \{ |(T\phi, \psi)| \mid \|\phi\| = 1 = \|\psi\| \}$
$= \{ |(\psi, T\phi)| \mid \|\phi\| = 1 = \|\psi\| \} = \|T\|$.

* The operator T^* is called the adjoint of T.

* The adjoint mapping $T \mapsto T^*$ defines an involution ($(T^*)^* = T$) on $\mathcal{B}(\mathcal{H})$ which makes it into a C^*-algebra:

3.3. Thm: For all $T, S \in \mathcal{B}(\mathcal{H})$, $\alpha \in \mathbb{C}$

a) $(T+S)^* = T^* + S^*$
b) $(\alpha T)^* = \alpha^* T^*$
c) $(ST)^* = T^* S^*$
d) $T^{**} = T$
e) $\|T^* T\| = \|T\|^2$.

(Notations: $ST := S \circ T$ and $T^{**} := (T^*)^*$)

Pf. Exercise □

3.4. Definitions

An operator $T \in \mathcal{B}(\mathcal{H})$ is called

a) self-adjoint if $T^* = T$
b) unitary if $T^*T = 1 = TT^*$
($\Rightarrow$ unitary also in the general Hilbert-space-isomorphism-sense)
c) normal if $T^*T = TT^*$
d) projection if $T^2 = T$.
If also $R(T) = \ker(T)^\perp$, T is called an orthogonal projection.

* Note: P and Q in Theorem 2.16. are orthogonal projections.