2.14. Tensor products of Hilbert spaces

Motivation from QM:

Suppose particle 1 has a wave vector $\psi_1(\bar{x})$, $\bar{x} \in \mathbb{R}^3$, and particle 2 a wave vector $\psi_2(\bar{x})$, $\bar{x} \in \mathbb{R}^3$. Then $\psi_1, \psi_2 \in L^2(\mathbb{R}^3)$ and to each measurable sets $A_1, A_2 \subset \mathbb{R}^3$, the probabilities of finding the particles in these sets are

$$\mathbb{P}(\text{ particle } i \text{ in set } A_i) = \int_{A_1} d^3x\, |\psi_i(x)|^2 .$$

Q: Is there any natural way to describe the state of the joint system of these particles, with a wave function $\Psi(\bar{x}_1, \bar{x}_2)$, if we know that the particle positions are distributed independenthy?

$$\Leftrightarrow \mathbb{P}(\bar{x}_1 \in A_1 \text{ and } \bar{x}_2 \in A_2) = \mathbb{P}(\bar{x}_1 \in A_1)\, \mathbb{P}(\bar{x}_2 \in A_2)$$

$$\Leftrightarrow \int_{A_1 \times A_2} d^3\bar{x}_1\, d^3\bar{x}_2\, |\Psi(\bar{x}_1, \bar{x}_2)|^2 = \int d^3\bar{x}_1\, |\psi_1(\bar{x}_1)|^2 \times \int d^3\bar{x}_2\, |\psi_2(\bar{x}_2)|^2$$

By Fubini's thrm, it suffices to require that

$$|\Psi(\bar{x}_1, \bar{x}_2)|^2 = |\psi_1(\bar{x}_1)|^2\, |\psi_2(\bar{x}_2)|^2$$

and the simplest solution to this is given by

$$\Psi(\bar{x}_1, \bar{x}_2) := \psi_1(\bar{x}_1)\, \psi_2(\bar{x}_2), \quad \bar{x}_1, \bar{x}_2 \in \mathbb{R}^3$$

where $(\bar{x}_1, \bar{x}_2) \in \mathbb{R}^3 \times \mathbb{R}^3 \cong \mathbb{R}^6$ and $\Psi \in L^2(\mathbb{R}^6)$.

Then Ψ is called a tensor product of the vectors $\psi_1, \psi_2 \in L^2(\mathbb{R}^3)$ and we write

$$\Psi := \psi_1 \otimes \psi_2 \in L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3).$$

This gives a natural rule how to generate multiparticle states from one-particle states without introducing correlations between the particles.

In general, we introduce the following definition for N-fold tensor products of Hilbert spaces:

Definition (tensor product)

Suppose $N \in \mathbb{N}$, $N \geq 2$, and that to each $k = 1, \dots, N$, there are given a Hilbert space $\mathcal{H}_k$ and its ONB $(e_k^{(\ell)})_{\ell \in I_k}$ where I_k is an index set for the basis (hence, $\operatorname{card} I_k = \dim \mathcal{H}_k$).

Consider their product space $X := \prod_{k=1}^{N} \mathcal{H}_k$, and call a functional (= map) $T : X \to \mathbb{C}$ separately conjugate-linear and continuous, if for every $\psi \in X$ and $k = 1, \dots, N$, the map

$$\phi_k \longmapsto T(\psi_1, \dots, \underset{\text{slot } k}{\phi_k}, \dots, \psi_N)^*$$

from $\mathcal{H}_k \to \mathbb{C}$ is linear and continuous.

Collect to the set $\mathcal{H}$ all such functionals which satisfy:

$$\sum_{\underline{l}\in I} |T(e_{\underline{l}})|^2 < \infty$$

where $I := \prod_{k=1}^{N} I_k$ and $e_{\underline{l}} := (e_1^{(l_1)}, \dots, e_N^{(l_N)})$

for any $\underline{l} = (l_1, \dots, l_N) \in I$.

Then, by Hölder's inequality, for any $T, U \in \mathcal{H}$ we can define

$$\langle T | U \rangle := \sum_{\underline{l}\in I} T(e_{\underline{l}})^* U(e_{\underline{l}}) \in \mathbb{C}.$$

Proposition: $\mathcal{H}$ is a vector space and $(T,U) \mapsto \langle T|U\rangle$ is a scalar product on $\mathcal{H}$.

Proof: The definition implies that the map $\Phi : T \mapsto (T(e_{\underline{l}}))_{\underline{l}\in I}$ is a map from $\mathcal{H}$ to the Hilbert space $l_2(I)$ ($:= L^2(\mu)$ for μ = counting measure on I. See Ex. 2.2. for an example where I is countable).

The vector operations are defined by the rules

$$(\alpha T + \beta U)(\varkappa) := \alpha T(\varkappa) + \beta U(\varkappa), \quad \alpha, \beta \in \mathbb{C},\ T, U \in \mathcal{H},\ \varkappa \in \Xi$$

and thus the map Φ is linear.

Using these observations, it is straightforward to check that $\alpha T + \beta U \in \mathcal{H}$ for all $\alpha, \beta \in \mathbb{C}$, $T, U \in \mathcal{H}$, and that $\mathcal{H}$ is a complex vector space.

Since $l_2(I)$ is a Hilbert space and by definition $\langle T | U\rangle_{\mathcal{H}} = \langle \Phi(T) | \Phi(U)\rangle_{l_2(I)}$

it is easy to check that items "a) - d)" on p. 12 are satisfied. It only remains to prove that

$$\langle T | T \rangle = 0 \Rightarrow T = 0.$$

To prove this, let us first prove the following Lemma about representation of T in terms of $\Phi(T)$.

Lemma: Suppose $T \in \mathcal{H}$, and denote $\lambda_{\underline{\ell}} := \Phi(T)_{\underline{\ell}}$, $\underline{\ell} \in I$. Then for every $\underline{x} \in \underline{X}$ we have

$$(1) \qquad T(\underline{x}) = \sum_{\underline{\ell} \in I} \lambda_{\underline{\ell}} \prod_{k=1}^{N} \langle x_k | e_k^{(\ell_k)} \rangle_{\mathcal{H}_k}$$

where the sum is absolutely convergent.

Proof: Now $\underline{x} = (x_1, \ldots, x_N) \in \underline{X}$ and for each k $x_k \in \mathcal{H}_k$. Since $(e_k^{(\ell)})_{\ell \in I_k}$ is ONB for $\mathcal{H}_k$, it follows that $\alpha_k^{\ell_k} := \langle x_k | e_k^{(\ell_k)} \rangle_{\mathcal{H}_k}$ satisfy $\sum_{\ell_k \in I_k} |\alpha_k^{\ell_k}|^2 = \|x_k\|^2_{\mathcal{H}_k} < \infty$.

Thus only countably many of α_k^{ℓ} are non-zero, and there is an increasing sequence of finite subsets $I_k^{(n)} \subset I_k$, $n \in \mathbb{N}$, such that

$$x_k = \lim_{n \to \infty} x_k^{(n)} \text{ in } \mathcal{H}_k\text{-norm},$$

where $x_k^{(n)} := \sum_{\ell_k \in I_k^{(n)}} \langle e_k^{(\ell_k)} | x_k \rangle e_k^{(\ell_k)}$

$$= \sum_{\ell_k \in I_k^{(n)}} (\alpha_k^{\ell_k})^* e_k^{(\ell_k)}.$$

Now for any $\underline{n} \in \mathbb{N}^N$ using multi-linearity of T we have

$$T(x_1^{(n_1)}, \ldots, x_N^{(n_N)}) = \sum_{\ell_1 \in I_1^{(n_1)}} \cdots \sum_{\ell_N \in I_N^{(n_N)}} \prod_{k=1}^{N} ((\alpha_k^{\ell_k})^*)^* \, T(e_1^{(\ell_1)}, \ldots, e_N^{(\ell_N)})$$

$$= \sum_{\underline{\ell} \in \prod_{k=1}^{N} I_k^{(n_k)}} \prod_{k=1}^{N} \alpha_k^{\ell_k} \cdot T(e_{\underline{\ell}}) = \sum_{\underline{\ell} \in \prod_{k=1}^{N} I_k^{(n_k)}} \lambda_{\underline{\ell}} \prod_{k=1}^{N} \langle x_k | e_k^{(\ell_k)} \rangle_{\mathcal{H}_k}$$

By separate continuity of T, we thus have

$$T(\varkappa) = T(\varkappa_1, \dots, \varkappa_N) = \lim_{n_1 \to \infty} \cdots \lim_{n_N \to \infty} \sum_{\underline{l} \in I} \mathbb{1}_{\{\underline{l} \in \prod_{k=1}^{N} I_k^{(n_k)}\}}$$
$$\times \lambda_{\underline{l}} \prod_{k=1}^{N} \langle \varkappa_k | e_k^{(l_k)} \rangle$$

Since $T \in \mathcal{H}$, we have $\sum_{\underline{l} \in I} |\lambda_{\underline{l}}|^2 < \infty$,

and, on the other hand,

$$\sum_{\underline{l} \in I} \left| \prod_{k=1}^{N} \langle \varkappa_k | e_k^{(l_k)} \rangle \right|^2 = \prod_{k=1}^{N} \left(\sum_{l_k \in I_k} |\langle \varkappa_k | e_k^{(l_k)} \rangle|^2 \right)$$
$$= \prod_{k=1}^{N} \|\varkappa_k\|^2_{\mathcal{H}_k} < \infty,$$

Hölder's inequality implies that $\sum_{\underline{l} \in I} |\lambda_{\underline{l}}| \left| \prod_{k=1}^{N} \langle \varkappa_k | e_k^{(l_k)} \rangle \right| < \infty$.

Thus we can apply Lebesgue's dominated convergence theorem to the counting measure over I. Since $\mathbb{1}_{\{\underline{l} \in \prod_{k=1}^{N} I_k^{(n_k)}\}} \to 1$, we have

$$T(\varkappa) = \sum_{\underline{l} \in I} \lambda_{\underline{l}} \prod_{k=1}^{N} \langle \varkappa_k | e_k^{(l_k)} \rangle_{\mathcal{H}_k} \qquad \square$$

Now if $\langle T | T \rangle = 0 \Rightarrow \sum_{\underline{l} \in I} |\lambda_{\underline{l}}|^2 = 0$

$\Rightarrow \lambda_{\underline{l}} = 0 \ \forall \underline{l} \in I$. Thus by the Lemma $\forall \varkappa \in \mathfrak{X}$,

$T(\varkappa) = 0$, and hence $T = 0$ (= null of $\mathcal{H}$)

$\therefore \langle \cdot | \cdot \rangle$ defines a scalar product on $\mathcal{H}$. $\square$

<u>Theorem</u>: $\mathcal{H}$ is a Hilbert space isomorphic to $l_2(I)$, $\mathcal{H} \simeq l_2(I)$.

<u>Proof</u>: For any $\lambda \in l_2(I)$, we can use formula (1) to define $T(\varkappa)$ for every $\varkappa \in \mathfrak{X}$. This gives a map $T : \mathfrak{X} \to \mathbb{C}$, and as above one can check that the defining sum is absolutely convergent, and hence apply Fubini's theorem

to check that the map is separately conj.-linear. It also follows using Hölder's inequality that

$$|T(x)| \le \|\lambda\|_{l_2(I)} \prod_{k=1}^{N} \|x_k\|_{\mathcal{H}_k}.$$

This shows that each of the k separate maps are bounded, hence continuous. Thus $T \in \mathcal{H}$, and let us denote the map $\lambda \mapsto T$ by $\tilde{\Phi} : l_2(I) \to \mathcal{H}$. Since then for any $\ell \in I$,

$$\tilde{\Phi}(\lambda)(e_{\underline{\ell}}) = \sum_{\underline{\ell}' \in I} \lambda_{\underline{\ell}'} \prod_{k=1}^{N} \underbrace{\langle e_k^{(\ell_k)} | e_k^{(\ell'_k)} \rangle}_{\overset{ONB}{=} \mathbb{1}\{\ell_k = \ell'_k\}}$$

$$= \sum_{\underline{\ell}' \in I} \lambda_{\underline{\ell}'} \mathbb{1}_{\{\underline{\ell}' = \underline{\ell}\}} = \lambda_{\underline{\ell}}$$

we have $\Phi \circ \tilde{\Phi} = id_{l_2(I)}$. On the other hand, by the Lemma above, $\tilde{\Phi} \circ \Phi = id_{\mathcal{H}}$, so we can conclude that Φ is an invertible map. As it is also linear, and preserves scalar products, $\langle T | U \rangle_{\mathcal{H}} = \langle \Phi(T) | \Phi(U) \rangle_{l_2(I)}$

Φ is a unitary transformation between $\mathcal{H}$ and $l_2(I)$. It forms a Hilbert-space isomorfism between these two spaces, in particular, implying that $\mathcal{H}$ is complete. □
(see Ex. 3.5. and Wikipedia / Funct. A. books for more details).

<u>Definition</u>: If $\phi_k \in \mathcal{H}_k$, $k = 1, \dots, N$, are given, their <u>tensor product</u> is the functional

$$T = \bigotimes_{k=1}^{N} \phi_k \in \mathcal{H} \quad \text{defined by the formula}$$

$$T(x) = \prod_{k=1}^{N} \langle x_k | \phi_k \rangle_{\mathcal{H}_k} \quad \forall x \in \underline{X}.$$

(To check that $T \in \mathcal{H}$, note that by Cauchy-Schw.

$$|T(x)| \leq \prod_{k=1}^{N} \|x_k\| \cdot \prod_{k=1}^{N} \|\phi_k\|$$

and thus it is sep. continuous and obviously conj.-multilin. Also, for $\underline{l} \in I$

$$T(e_{\underline{l}}) = \prod_{k=1}^{N} \langle e_k^{(l_k)} | \phi_k \rangle$$

$$\Rightarrow \sum_{\underline{l} \in I} |T(e_{\underline{l}})|^2 = \prod_{k=1}^{N} \Big(\sum_{l_k \in I_k} |\langle e_k^{(l_k)} | \phi_k \rangle|^2 \Big)$$

$$= \prod_{k=1}^{N} \|\phi_k\|^2_{\mathcal{H}_k} < \infty. \quad \therefore \; T \in \mathcal{H}. \Big)$$

* The representation (1) can now be summarized in the formula:

(2) If $T \in \mathcal{H}$ and $x \in \bar{X}$, we have

$$T(x) = \langle \bigotimes_{k=1}^{N} x_k \,|\, T \rangle_{\mathcal{H}} .$$

__Proof.__ By definition, the r.h.s. is

$$\langle \bigotimes_{k=1}^{N} x_k \,|\, T \rangle = \sum_{\underline{l} \in I} \langle \bigotimes_{k=1}^{N} x_k | e_{\underline{l}} \rangle^* T(e_{\underline{l}})$$

$$= \sum_{\underline{l} \in I} \prod_{k=1}^{N} \langle e_k^{(l_k)} | x_k \rangle^* T(e_{\underline{l}})$$

$$= \sum_{\underline{l} \in I} \underbrace{T(e_{\underline{l}})}_{=\lambda_{\underline{l}}} \prod_{k=1}^{N} \langle x_k | e_k^{(l_k)} \rangle \overset{(1)}{=} T(x). \;\square$$

* $\Rightarrow$ If $\phi, x \in \bar{X}$, then

(3)

$$\langle \bigotimes_{k=1}^{N} x_k \,|\, \bigotimes_{k=1}^{N} \phi_k \rangle_{\mathcal{H}} = \Big(\bigotimes_{k=1}^{N} \phi_k\Big)(x)$$

$$= \prod_{k=1}^{N} \langle x_k | \phi_k \rangle_{\mathcal{H}_k}$$

* Note that the map $x \mapsto \bigotimes_{k=1}^{N} x_k$ is __not__ one-to-one. For instance, if just one of the x_k is a null vector $\Rightarrow \bigotimes_{k=1}^{N} x_k$ = null functional.

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* At the moment, it looks as if the definition of $\mathcal{H}$ depends on the choice of the ONBs $(e_k^{(l)})$. The following result shows that, in fact, both $\mathcal{H}$ and the value of the scalar product are independent of the choice.

Proposition: Let $(f_k^{(l)})_{l\in I_k}$, $k=1,\dots,N$, be another collection of ONBs.

If T is a separately conj.-multilinear and continuous functional, then

$$(*) \qquad \sum_{l\in I} |T(e_l)|^2 = \sum_{l\in I} |T(f_l)|^2$$

also when one of the sums is infinite.

In addition, if $T, U \in \mathcal{H}$, we have

$$(**) \qquad \sum_{l\in I} T(e_l)^* U(e_l) = \sum_{l\in I} T(f_l)^* U(f_l).$$

Proof: If both sums in $(*)$ $=\infty$, there is nothing to prove. Suppose thus first that $\sum_{l\in I} |T(e_l)|^2 < \infty \Rightarrow T \in \mathcal{H}$ (as def. above).

For any finite $F \subset I$, the lin. komb.

$$T_F := \sum_{l\in F} T(f_l) \bigotimes_{k=1}^{N} f_k^{(l_k)} \in \mathcal{H}.$$

By the earlier results, then

$$0 \le \|T - T_F\|^2 = \langle T - T_F | T - T_F \rangle_{\mathcal{H}}$$

$$= \langle T | T \rangle - \sum_{l\in F} T(f_l)^* \langle \otimes f_k^{(l_k)} | T \rangle$$

$$- \sum_{l\in F} T(f_l) \langle T | \otimes f_k^{(l_k)} \rangle + \sum_{l',l\in F} T(f_{l'})^* T(f_l) \langle \otimes f_k^{(l'_k)} | \otimes f_k^{(l_k)} \rangle$$

$$= \langle T | T \rangle - \sum_{l\in F} T(f_l)^* T(f_l) - \sum_{l\in F} T(f_l) T(f_l)^*$$

$$+ \sum_{l',l\in F} T(f_{l'})^* T(f_l) \prod_{k=1}^{N} \underbrace{\langle f_k^{(l'_k)} | f_k^{(l_k)} \rangle_{\mathcal{H}_k}}_{\overset{ONB}{=} 1\{l'_k = l_k\}}$$

Since $\langle T|T\rangle_{\mathcal{H}} = \sum_{\ell\in I} |T(e_\ell)|^2 < \infty$

$$\Rightarrow \quad (2-1)\sum_{\ell\in F} |T(f_\ell)|^2 = \sum_{\ell\in F} |T(f_\ell)|^2$$
$$\leq \langle T|T\rangle_{\mathcal{H}} = \sum_{\ell\in I} |T(e_\ell)|^2 < \infty$$

As the set F is arbitrary finite subset, this shows that

$$\sum_{\ell\in I} |T(f_\ell)|^2 = \sup_{F\subset I,\text{finite}} \sum_{\ell\in F} |T(f_\ell)|^2$$
$$\leq \sum_{\ell\in I} |T(e_\ell)|^2 .$$

In particular, then $\sum_{\ell\in I} |T(f_\ell)|^2 < \infty$, so the same argument can be performed in the space $\mathcal{H}'$ defined using (f_ℓ) as the basis.

$$\Rightarrow \quad \sum_{\ell\in I} |T(e_\ell)|^2 \leq \sum_{\ell\in I} |T(f_\ell)|^2 .$$

Thus, together these inequalities yield

$$\sum_{\ell\in I} |T(e_\ell)|^2 = \sum_{\ell\in I} |T(f_\ell)|^2 \quad \text{as claimed.}$$

Since the roles of (e_ℓ) and (f_ℓ) can be interchanged, this also proves the statement starting from assumption $\sum_{\ell\in I} |T(f_\ell)|^2 < \infty$.

Thus the definition of the <u>set</u> $\mathcal{H}$ does not depend on the choice of ONBs. Hence, if $T, U \in \mathcal{H}$ equality holds in $(*)$ and both sums are finite.

Using the polarization identity, we may write

$$T(f_\ell)^* U(f_\ell) = \frac{1}{4}\sum_{n=0}^{3} i^{-n} |\underbrace{T(f_\ell) + i^n U(f_\ell)}_{= (T+i^n U)(f_\ell)}|^2$$

Hence, $(*)$ implies that also $(**)$ holds. $\square$

* Therefore, the definition of the Hilbert space $\mathcal{H}$ will remain unchanged if one starts with some other choice of ONBs. Thus this choice can be dropped from the notation, and we call $\mathcal{H}$ the tensor product of the Hilbert spaces $\mathcal{H}_k$, and denote

$$\mathcal{H} = \bigotimes_{k=1}^{N} \mathcal{H}_k .$$

* The following usufel result is now a straightforward consequence:

Thrm: Suppose $(e_k^{(\ell)})_{\ell \in I_k}$ is some ONB for the Hilbert space $\mathcal{H}_k$, $k=1,\dots,N$. then the functionals

$$E_{\underline{\ell}} := \bigotimes_{k=1}^{N} e_k^{(\ell_k)} \quad , \quad \underline{\ell} \in \underline{I} ,$$

form an ONB for $\mathcal{H} = \bigotimes_{k=1}^{N} \mathcal{H}_k$.

Proof: By definition, each $E_{\underline{\ell}} \in \mathcal{H}$. If $\underline{\ell}' \neq \underline{\ell}$, we have $\langle E_{\underline{\ell}'} | E_{\underline{\ell}} \rangle_{\mathcal{H}} \overset{(3)}{=} \prod_{k=1}^{N} \underbrace{\langle e_k^{(\ell_k')} | e_k^{(\ell_k)} \rangle}_{= \mathbb{1}\{\ell_k' = \ell_k\}}$

$= \mathbb{1}\{\underline{\ell}' = \underline{\ell}\}$

Thus $(E_{\underline{\ell}})_{\underline{\ell} \in \underline{I}}$ is an orthonormal set.

If $T \in \mathcal{H}$, we have $\lambda_{\underline{\ell}} := T(e_{\underline{\ell}}) \overset{(2)}{=} \langle \bigotimes_{k=1}^{N} e_k^{(\ell_k)} | T \rangle_{\mathcal{H}}$
$= \langle E_{\underline{\ell}} | T \rangle_{\mathcal{H}}$.
Thus $\|T\|^2 = \sum_{\underline{\ell} \in \underline{I}} |\lambda_{\underline{\ell}}|^2 < \infty$, implies that there is an increasing sequence $\underline{I}^{(n)} \subset \underline{I}$ such that each $\underline{I}^{(n)}$ is finite and

$$\|T\|^2 = \lim_{n\to\infty} \sum_{\underline{\ell} \in \underline{I}^{(n)}} |\lambda_{\underline{\ell}}|^2 .$$

As earlier, we may then define $T_n \in \operatorname{span}_{\underline{\ell} \in \underline{I}} (E_{\underline{\ell}})$

by the formula $T_n := \sum_{\ell \in I^{(n)}} \lambda_\ell E_\ell$,

and, as above, conclude that

$$0 \leq \|T - T_n\|^2 = \|T\|^2 - \sum_{\ell \in I^{(n)}} |\lambda_\ell|^2 \xrightarrow{n\to\infty} 0.$$

Thus $T_n \to T$ in norm, and hence $T \in \overline{\operatorname{span}(E_\ell)}_{\ell \in I}$. This proves that $(E_\ell)_{\ell \in I}$ is indeed an ONB for $\mathcal{H}$. $\square$

Corollary 1:
$$\dim\Big(\bigotimes_{k=1}^{N} \mathcal{H}_k\Big) = \prod_{k=1}^{N} \dim \mathcal{H}_k.$$

Proof. Since $\operatorname{card}(I) = \operatorname{card}\Big(\prod_{k=1}^{N} I_k\Big) = \prod_{k=1}^{N} \big(\overbrace{\operatorname{card} I_k}^{=\dim \mathcal{H}_k}\big)$, and by the previous result $\mathcal{H}$ has an ONB indexed by I, we have $\dim \mathcal{H} = \operatorname{card}(I) = \prod_{k=1}^{N} \dim \mathcal{H}_k$ $\square$

Corollary 2: The span of all tensor products $\bigotimes_{k=1}^{N} u_k$, $u \in \mathcal{B}$, is dense is $\mathcal{H}$.

Proof: Since every $\bigotimes_{k=1}^{N} u_k \in \mathcal{H}$, also their span is a subset of the vector space $\mathcal{H}$. Since its subset, $\operatorname{span}(E_\ell)_{\ell \in I}$, is already dense in $\mathcal{H}$, also the full span is dense in $\mathcal{H}$. $\square$

* This last result shows that $\mathcal{H}$ indeed gives an explicit realization of the more common abstract definition of tensor product spaces. (Cf. The Appendix in Hall's book. Using its notations, we have here $\mathcal{H} = \hat{\bigotimes} \mathcal{H}_k$, as $\mathcal{H}$ is already complete.)

* This construction merely gives finite tensor products of Hilbert spaces (here $2 \leq N < \infty$). The completion step in the case of infinitely many Hilbert spaces, as well as when considering even finite tensor products of Banach spaces, is not obvious (and maybe not even unique).

Example 1: A functional $T \in \mathbb{C}^2 \otimes \mathbb{C}^3$

can thus be written in the basis $\hat{e}_i \otimes \hat{e}_j$, $i = 1, 2$, $j = 1, 2, 3$ as

$$T = \sum_{i=1}^{2} \sum_{j=1}^{3} T_{ij} \, \hat{e}_i \otimes \hat{e}_j .$$

where $T_{ij} = \langle \hat{e}_i \otimes \hat{e}_j | T \rangle = T(\hat{e}_i, \hat{e}_j)$.

The correspondence $T \leftrightarrow T_{ij}$ is analogous to that of between linear maps and matrices. As

$$\langle T | U \rangle = \sum_{i=1}^{2} \sum_{j=1}^{3} T_{ij}^{*} \, U_{ij} ,$$ this makes

$\mathbb{C}^2 \otimes \mathbb{C}^3$ isomorphic to $\mathbb{C}^{2\times 3} = \mathbb{C}^6$ as a Hilbert space.

Example 2: As in the beginning of the section, let $\psi_1, \psi_2 \in L^2(\mathbb{R}^3) =: \mathcal{H}_1$

Then $\forall \, (\phi_1, \phi_2) \in L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$, we have

$$(\psi_1 \otimes \psi_2)(\phi_1, \phi_2) = \langle \phi_1 | \psi_1 \rangle \langle \phi_2 | \psi_2 \rangle .$$ There

turns out to be a unique unitary map $\Psi : \mathcal{H}_1 \otimes \mathcal{H}_1 \to L^2(\mathbb{R}^3 \times \mathbb{R}^3)$

for which $\Psi(\psi \otimes \phi)(\bar{x}_1, \bar{x}_2) = \psi(\bar{x}_1)\,\phi(\bar{x}_2)$.

This is consistent, since we indeed have

$$\langle \phi_1 \otimes \phi_2 | \psi_1 \otimes \psi_2 \rangle = \langle \phi_1 | \psi_1 \rangle \langle \phi_2 | \psi_2 \rangle$$

$$= \int d^3\bar{x}_1 \, \phi_1(\bar{x}_1)^* \psi_1(\bar{x}_1) \int d^3\bar{x}_2 \, \phi_2(\bar{x}_2)^* \psi_2(\bar{x}_2)$$

$$\overset{\text{Fubini}}{=} \int d^3\bar{x}_1 \, d^3\bar{x}_2 \, (\phi_1(\bar{x}_1)\phi(\bar{x}_2))^* \psi_1(\bar{x}_1)\psi_2(\bar{x}_2) = \langle \Psi(\phi_1 \otimes \phi_2) | \Psi(\psi_1 \otimes \psi_2) \rangle$$