2.5. The mother of all Hilbert spaces:

Let μ be a positive measure on $\mathbb{X}$.
(For instance, $\mathbb{X} = \mathbb{R}^d$, $d\mu = dx$ = Lebesgue measure.)
Define

$$L^2_{pre}(\mu) = \left\{ \psi : \mathbb{X} \to \mathbb{C} \text{ measurable} \,\middle|\, \int_{\mathbb{X}} \mu(dx) \, |\psi(x)|^2 < \infty \right\}$$

and let

$$(\phi, \psi) = \int_{\mathbb{X}} \mu(dx) \, \phi(x)^* \psi(x) .$$

Then $(\cdot,\cdot)$ satisfies a) – d), but not e): Let $\psi' \sim \psi :\Leftrightarrow \psi' = \psi$ a.e.
$\Leftrightarrow \exists E \subset \mathbb{X}$, measurable, s.t. $\mu(E) = 0$ and $\forall x \notin E : \psi'(x) = \psi(x)$.
Then $\psi \sim 0 \Rightarrow (\psi, \psi) = 0$.
This, however, can be remedied easily:
"$\sim$" is an equivalence relation
$\Rightarrow$ can define $L^2(\mu) = L^2_{pre}(\mu)/\sim$
$= \{$ set of equivalence classes w.r.t. $\sim \}$
Since (ϕ, ψ) remains invariant if ϕ or ψ is modified on a set of measure zero, it yields a well-defined mapping
$L^2(\mu) \times L^2(\mu) \to \mathbb{C}$ (the integral is finite by Hölder's ineq.)
$L^2(\mu)$ is also a complex vector space, and

$$\sqrt{(\psi,\psi)} = \sqrt{\int_{\mathbb{X}} \mu(dx) \, |\psi(x)|^2} = L^p\text{-norm for } p = 2.$$

Thus by the completeness of L^p-spaces,(*) $L^2(\mu)$ is complete in $\|\cdot\| = \sqrt{(\cdot,\cdot)}$.

[$\therefore$ $L^2(\mu)$ is a Hilbert space.]

(*) See, for instance, Rudin: RCA, Th. 3.11.

* For μ = Lebesgue, we write $L^2(\mu) = L^2(\mathbb{R}^d)$.
* The following result (Rudin, RCA: Th. 3.12.) is useful about norm-convergent sequences:

Th. If $1 \le p \le \infty$ and $f_n \to f$ in $L^p(\mu)$-norm, then there is a subsequence $(n_k)_{k\in\mathbb{N}}$ s.t.
$$f_{n_k}(x) \xrightarrow{k\to\infty} f(x) \quad \text{a.e. } x \in X.$$

Thus, for instance, any norm-convergent sequence in $L^2(\mu)$ has a pointwise a.e. convergent subseq.

Elementary properties of Hilbert spaces

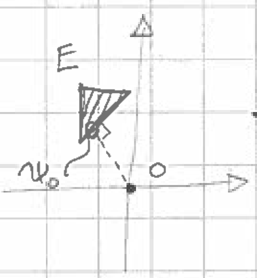
2.5. Thm: $\psi \mapsto \|\psi\|$ is a continuous mapping $\mathcal{H} \to \mathbb{R}$. $\forall \phi \in \mathcal{H}$ both $\psi \mapsto (\phi, \psi)$ and $\psi \mapsto (\psi, \phi)$ are continuous $\mathcal{H} \to \mathbb{C}$.
Pf: Chauchy-Schwarz (Rudin, RCA, Th. 4.6.) $\square$

2.6. thm: If $M \subset \mathcal{H}$ is subspace, then its (norm-) closure $\overline{M}$ is a closed subspace. If M is also closed, it is a Hilbert space with the induced scalar product.
Pf: Rudin, RCA, Sect. 4.7. $\square$

2.7. Definition: Let V be a vector space. $E \subset V$ is called convex, if

$$\phi, \psi \in E, \quad t \in [0,1] \Rightarrow (1-t)\psi + t\phi \in E.$$

that is, if the line connecting any $\phi, \psi \in E$ belongs to E.



Thm: If $E \subset \mathcal{H}$ is non-empty, closed, and convex, then $\exists! \, \psi_0 \in E$ s.t.
$$\|\psi_0\| = \inf\{\|\psi\| \mid \psi \in E\}$$
$\Leftrightarrow$ "ψ_0 is a norm-minimizer"

Proof: For any $\psi, \phi \in \mathcal{H}$ it holds that

$$(PL) \quad \|\psi+\phi\|^2 + \|\psi-\phi\|^2 = 2\|\psi\|^2 + 2\|\phi\|^2$$

(the "cross-terms" cancel; this identity is called the parallelogram law.)

Let $\delta := \inf\{\|\psi\| \mid \psi \in E\} \geq 0$. Consider $\psi, \phi \in E$, and apply (PL) to $\frac{1}{2}\psi$ and $\frac{1}{2}\phi$

$$\Rightarrow \frac{1}{4}\|\psi-\phi\|^2 = \frac{1}{2}\|\psi\|^2 + \frac{1}{2}\|\phi\|^2 - \left\|\frac{\psi+\phi}{2}\right\|^2$$

Since E is convex $\Rightarrow \frac{\psi+\phi}{2} \in E \Rightarrow \delta^2 \leq \left\|\frac{\psi+\phi}{2}\right\|^2$

Thus $\|\psi-\phi\|^2 \leq 2\|\psi\|^2 + 2\|\phi\|^2 - 4\delta^2$. $\forall \psi, \phi \in E$.
Therefore, if $\|\psi\| = \delta = \|\phi\| \Rightarrow \|\psi-\phi\|^2 \leq 0$
$\Rightarrow \psi = \phi$. This proves that any minimizer is unique.

For existence: By definition of δ, $\exists$ sequence $(\psi_n)_{n\in\mathbb{N}}$ s.t. $\psi_n \in E$ $\forall n$ and $\delta = \lim_{n\to\infty} \|\psi_n\|$.

Since then $\forall m,n$: $\|\psi_n - \psi_m\|^2$
$\leq 2\|\psi_n\|^2 + 2\|\psi_m\|^2 - 4\delta^2 \xrightarrow[n,m\to\infty]{} 0$,

(ψ_n) is a Cauchy sequence. As $\mathcal{H}$ is complete, $\exists\, \phi \in \mathcal{H}$ s.t. $\phi = \lim_{n\to\infty} \psi_n$, and in fact

then $\phi \in E$ since E is closed. $\|\cdot\|$ is continuous $\Rightarrow \delta = \lim_{n\to\infty} \|\psi_n\| = \|\lim \psi_n\| = \|\phi\|$.

Thus ϕ is a minimizer. $\square$

* Note that the proof relies heavily on (PL) which is a consequence of $\|\psi\|^2 = (\psi, \psi)$, i.e., of existence of scalar product.

2.8. Defn. (internal direct sum in $\mathcal{H}$)

If $M_1, M_2 \subset \mathcal{H}$ are <u>closed</u> subspaces and $\psi \perp \phi \quad \forall \psi \in M_1, \phi \in M_2$.

$$M_1 \oplus M_2 = \{\psi_1 + \psi_2 \mid \psi_i \in M_i ; i = 1,2\} \subset \mathcal{H}.$$

* Then $M_1 \cap M_2 = \{0\}$. (If $\psi \in M_1 \cap M_2 \Rightarrow \psi \perp \psi \overset{2.1.e)}{\Rightarrow} \psi = 0$.)

* $M_1 \oplus M_2$ is a closed subspace.
$\Rightarrow$ Further iteration is possible.

Example: Let $e_i \in \mathbb{C}^3$ be the unit vectors defined by $(e_i)_j = \delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$
(= Kronecker delta.)
Let $M_i = \mathrm{span}(\{e_i\}) = \{\alpha e_i \mid \alpha \in \mathbb{C}\}$
$\Rightarrow \mathbb{C}^3 = M_1 \oplus M_2 \oplus M_3$.

2.9. Remark: For any $E \subset \mathcal{H}$, its orthogonal complement $E^\perp$ is a closed subspace.
(Linearity is obvious. It is closed as
$$E^\perp = \bigcap_{\phi \in E} \{\phi\}^\perp, \text{ and } \{\phi\}^\perp = f_\phi^{\leftarrow}(\{0\})$$
is closed, since $f_\phi : \mathcal{H} \to \mathbb{C}$ defined by $f_\phi(\psi) = (\phi, \psi)$ is continuous. (see 2.5.))

2.10. <u>Defn:</u> If $M \subset \mathcal{H}$ is a <u>closed</u> subspace, and $\psi \in \mathcal{H}$, the set $E = \{\psi - \phi \mid \phi \in M\}$ is non-empty, closed and convex. Therefore, by 2.7. $\exists! \, \phi_\psi \in M$ s.t. $\|\psi - \phi_\psi\| \leq \|\psi - \phi\|$, $\forall \phi \in M$. Let $P : \mathcal{H} \to \mathcal{H}$ denote the mapping $\psi \mapsto \phi_\psi$. P is obviously a <u>projection</u> ($P^2 = P$) onto M.

Since $M^\perp$ is also a closed subspace, we can construct a projection Q onto $M^\perp$.

The following result proves that P and Q are, in fact, orthogonal projections.

2.11. **Thm**: Let $M \subset \mathcal{H}$ be closed subspace. Then

(i) $\mathcal{H} = M \oplus M^\perp$

(ii) The projections P onto M and Q onto $M^\perp$ are linear, and $P+Q=1$.

(iii) $\|\psi\|^2 = \|P\psi\|^2 + \|Q\psi\|^2 \quad \forall \psi \in \mathcal{H}$.

Pf: If $\psi \in M$, $\phi \in M^\perp$, then by definition of $M^\perp$ we have $(\phi,\psi)=0$. Since $M, M^\perp$ are also closed subspaces, $M \oplus M^\perp \subset \mathcal{H}$.
Let $\psi_0 \in \mathcal{H}$ be arbitrary, and consider $\phi_0 = \psi_0 - P\psi_0$. By definition of P, then $\forall \phi \in M$: $\|\phi_0\| \leq \|\psi_0 - \phi\|$
$= \|\psi_0 - P\psi_0 + P\psi_0 - \phi\| = \|\phi_0 + P\psi_0 - \phi\|$.
$\Rightarrow \forall \psi \in M: \|\phi_0\| \leq \|\phi_0 + \psi\|$.
Thus if $\psi \in M$, also $\lambda\psi \in M \quad \forall \lambda \in \mathbb{C}$, and by 2.3.(iii) then $(\psi, \phi_0) = 0$.
Therefore, $\phi_0 \in M^\perp$ and $\psi_0 = P\psi_0 + \phi_0$, where $P\psi_0 \in M$. This proves (i).
In fact, $\phi_0 = Q\psi_0$: If $\phi \in M^\perp$, then

$$\|\psi_0 - \phi\|^2 = \|\underbrace{P\psi_0}_{\in M} + \underbrace{\phi_0 - \phi}_{\in M^\perp}\|^2 \underset{(P\psi_0, \phi_0-\phi)=0}{=} \|P\psi_0\|^2 + \|\phi_0 - \phi\|^2$$

$$\geq \|P\psi_0\|^2 = \|\psi_0 - \phi_0\|^2.$$

Therefore, $\psi_0 = P\psi_0 + Q\psi_0$ and $1 = P+Q$.
Since $P\psi \perp Q\psi$, this implies also (iii).
Thus only the linearity in (ii) remains to be proven. For this, let $\psi, \phi \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$. By $1 = Q+P$, then

$$\psi = P\psi + Q\psi, \quad \phi = P\phi + Q\phi$$

and $\alpha\psi + \beta\phi = P(\alpha\psi + \beta\phi) + Q(\alpha\psi + \beta\phi)$.
However, then also

$$\alpha P\psi + \beta P\phi + \alpha Q\psi + \beta Q\phi = \alpha\psi + \beta\phi$$

and thus

$$M \ni P(\alpha\psi + \beta\phi) - \alpha P\psi - \beta P\phi$$
$$= -Q(\alpha\psi + \beta\phi) + \alpha Q\psi + \beta Q\phi \in M^\perp$$

and $M \cap M^\perp = \{0\}$ implies

$$P(\alpha\psi + \beta\phi) = \alpha P\psi + \beta P\phi,$$
$$Q(\alpha\psi + \beta\phi) = \alpha Q\psi + \beta Q\phi. \quad \square$$

2.12. * If $E \subset \mathcal{H}$ is non-empty
$$\operatorname{span} E = \bigcup_{n=1} \left\{ \sum_{i=1}^{n} \alpha_i \psi_i \;\middle|\; \alpha \in \mathbb{C}^n,\ \psi_i \in E\ \forall i \right\}.$$

* $E \subset \mathcal{H}$ is an orthonormal set,
if a) $e', e \in E$, $e' \neq e \Rightarrow (e', e) = 0$
b) $\|e\| = 1 \quad \forall e \in E$.

* A maximal orthonormal set in $\mathcal{H}$ is called an orthonormal basis (ONB) for the following reason:

Thm. Consider an orthonormal set $E = \{ e_i \mid i \in I \}$ in $\mathcal{H}$, indexed by an index set I. Then

$$E \text{ is maximal} \Leftrightarrow \overline{\operatorname{span} E} = \mathcal{H}.$$

If E is maximal,

a) $\forall \psi \in \mathcal{H}: \quad \|\psi\|^2 = \sum_{i \in I} |(e_i, \psi)|^2$

b) $\forall \phi, \psi \in \mathcal{H}: \quad (\phi, \psi) = \sum_{i \in I} (\phi, e_i)(e_i, \psi)$.

Pf: Rudin, RCA, 4.18. □

* If $\mathcal{H}$ has a finite ONB, it is finite-dimensional.

* If $\mathcal{H}$ has a countable ONB, it is called separable.

* $L^2(\mathbb{R}^d)$ is separable for all $1 \leq d < \infty$.

* There are also non-separable Hilbert spaces, for which I is uncountable. However, even then, for any $\psi \in \mathcal{H}$, the set $\{ i \in I \mid (e_i, \psi) \neq 0 \}$ is countable, and a) holds.

Direct sums and tensor products of Hilbert spaces

2.13. External direct sums

Let $\mathcal{H}_i$, $i \in I$, be a family of Hilbert spaces, where $I \neq \emptyset$ is some index set. Consider the following subset of the product space $\prod_{i \in I} \mathcal{H}_i$;

$$\mathcal{H} := \{ (\psi_i)_{i \in I} \mid \sum_{i \in I} \|\psi_i\|^2 < \infty \}$$

For $\Psi = (\psi_i)$ and $\Phi = (\phi_i)$ in $\mathcal{H}$, we define $\alpha\Psi$ and $\Psi + \Phi$ componentwise:

a) $(\alpha\Psi)_i := \alpha\psi_i \quad \forall i \in I, \ \alpha \in \mathbb{C}$

b) $(\Psi + \Phi)_i := \psi_i + \phi_i \quad \forall i$

Since $\|\alpha\psi_i\|^2 = |\alpha|^2 \|\psi_i\|^2$ and $\|\psi_i + \phi_i\|^2 \leq 2(\|\psi_i\|^2 + \|\phi_i\|^2)$ (by Hölder's inequality) then $\alpha\Psi, \Psi + \Phi \in \mathcal{H}$.
Also, then the set $I(\Psi) := \{ i \in I \mid \Psi_i \neq 0 \}$ is countable for all $\Psi \in \mathcal{H}$, and thus

$$\sum_{i \in I} \|\psi_i\| \|\phi_i\| = \sum_{i \in I(\Psi) \cup I(\Phi)} \|\psi_i\| \|\phi_i\| \overset{\text{Hölder}}{\leq} \sqrt{\sum_{i \in I(\psi)} \|\psi_i\|^2 \sum_{i \in I(\Phi)} \|\phi_i\|^2}$$

is finite, and therefore

$$((\Psi, \Phi)) := \sum_{i \in I} (\psi_i, \phi_i)_{\mathcal{H}_i}$$

defines a map $\mathcal{H} \times \mathcal{H} \to \mathbb{C}$, by Cauchy-Schwarz.

Theorem $\mathcal{H}$ (with $((\cdot, \cdot))$) is a Hilbert space.

Proof. Exercise □

* Notation: then we write $\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_i$.