12. Multiparticle quantum system, bosons and fermions

12.1. N-particle dynamics & translation invariance

Consider a system of N particles, enumerated by $n = 1, \dots, N$. Suppose that the state of particle n is determined by a wave vector in a Hilbert space h_n. As in Sec. 2.21., the wave vector of the joint system is then an element of the Hilbert space

$$\mathcal{H}_N := \bigotimes_{n=1}^{N} h_n .$$

The dynamics of the N-particle system is determined by a self-adjoint operator on $\mathcal{H}_N$. It is most often constructed as before, using $H_N = H_{0,N} + V_N$ where

a) $H_{0,N}$ = "free N-particle hamiltonian" $= \sum_{n=1}^{N} H_0^{(n,N)}$

where each $H_0^{(n,N)} := 1 \otimes 1 \otimes \dots \otimes \underset{\uparrow \text{ index } n}{H_0^{(n)}} \otimes 1 \otimes \dots \otimes 1$

and $H_0^{(n)}$ denotes the "free Hamiltonian" of particle n. Explicitly, the notations refers to the following construction:

1.1. Theorem: Suppose that for each $n = 1, \dots, N$ there is given a self-adjoint operator A_n on h_n. Define $\mathcal{H}_N := \bigotimes_{n=1}^{N} h_n$ and set

$$D(S) := \text{Span}\left\{ \bigotimes_{n=1}^{N} \psi_n \,\middle|\, \psi_n \in D(A_n)\ \forall n \right\} \subset \mathcal{H}_N .$$

Thus if $\psi \in D(S)$, we can find $M \in \mathbb{N}_+$, $\lambda \in \mathbb{C}^M$ and $\psi_{n,k} \in D(A_n)$, $k = 1, \dots, M$, $n = 1, \dots, N$, such that $\psi = \sum_{k=1}^{M} \lambda_k \bigotimes_{n=1}^{N} \psi_{n,k}$. Then setting, for $\psi \in D(S)$,

$$S\psi := \sum_{k=1}^{M} \lambda_k \bigotimes_{n=1}^{N} A_n \psi_{n,k}$$ yields a symmetric,

... densely defined operator on $\mathcal{H}_N$. Its closure $A := \overline{S}$ is a self-adjoint operator on $\mathcal{H}_N$ and we denote it by "$\bigotimes_{n=1}^{N} A_n$".

Proof: Suppose $\psi = \sum_{k=1}^{M} \lambda_k \bigotimes_{n=1}^{N} \psi_{n,k} = \sum_{k'=1}^{M'} \lambda'_{k'} \bigotimes_{n=1}^{N} \psi'_{n,k'}$

are two repre[se]ntations for $\psi \in D(S)$. For each $n = 1, \dots, N$, the span of $\{\psi_{n,k}\}_k \cup \{\psi'_{n,k'}\}_{k'}$, denoted M_n, is a finite subspace of h_n. Hence, M_n is closed and $h_n = M_n \oplus M_n^\perp$. Thus we can find an ONB for h_n such that the first $\dim M_n =: m_n$ vectors form an ONB for M_n and any other vector is orthogonal to M_n. Denote this ONB by $(e_n^{(\ell)})_{\ell \in I_n}$, I_n = index set. By Sec. 2.4. then

$$e(\ell) := \bigotimes_{n=1}^{N} e_n^{(\ell_n)}, \quad \ell \in I := \prod_{n=1}^{N} I_n, \text{ forms an ONB}$$

for $\mathcal{H}_N$. Set $a_{n,k}^{(\ell)} := \langle e_n^{(\ell)} | \psi_{n,k} \rangle$ and $b_{n,k'}^{(\ell)} := \langle e_n^{(\ell)} | \psi'_{n,k'} \rangle$.

$\Rightarrow \psi_{n,k} = \sum_{\ell=1}^{m_n} a_{n,k}^{(\ell)} e_n^{(\ell)}$ and $\psi'_{n,k'} = \sum_{\ell=1}^{m_n} b_{n,k'}^{(\ell)} e_n^{(\ell)}$.

Since clearly $M_n \subset D(A_n)$, we have with $I' := \prod_{n=1}^{N} \{1, \dots, m_n\}$

$$\sum_{k=1}^{M} \lambda_k \bigotimes_{n=1}^{N} A_n \psi_{n,k} = \sum_{\ell \in I'} \left(\sum_{k=1}^{M} \lambda_k \prod_{n=1}^{N} a_{n,k}^{(\ell_n)} \right) \bigotimes_{n=1}^{N} A_n e_n^{(\ell)}$$

$$\text{and } \sum_{k'=1}^{M'} \lambda'_{k'} \bigotimes_{n=1}^{N} A_n \psi'_{n,k'} = \sum_{\ell \in I'} \left(\sum_{k'=1}^{M'} \lambda'_{k'} \prod_{n=1}^{N} b_{n,k'}^{(\ell_n)} \right) \bigotimes_{n=1}^{N} A_n e_n^{(\ell)}$$

These are the same vector in $\mathcal{H}_N$, since for any $\ell \in I'$

$$(e(\ell), \psi) = \sum_{k=1}^{M} \lambda_k \prod_{n=1}^{N} a_{n,k}^{(\ell_n)} = \sum_{k'=1}^{M'} \lambda'_{k'} \prod_{n=1}^{N} b_{n,k'}^{(\ell_n)}.$$

On the other hand, $D(S)$ is dense: Suppose $\psi \in \mathcal{H}_N$ and $(e_n^{(\ell)})_{\ell \in I_n}$ is an ONB for h_n $\forall n$.
Define $e(\ell)$ and I as above, and set $a(\ell) := \langle e(\ell) | \psi \rangle$.
$\Rightarrow \sum_{\ell \in I} |a(\ell)|^2 = \|\psi\|^2 < \infty$. Consider some $\varepsilon > 0$. Suppose $\ell \in I$ is s.t. $a(\ell) \neq 0$.

For every n, $D(A_n)$ is dense in $\mathcal{H}_n \Rightarrow \exists f_n^{(\ell)} \in D(A_n)$ s.t. $\|e_n^{(\ell_n)} - f_n^{(\ell)}\| < \min(1, \varepsilon |a(\ell)|)$. Define then $f(\ell) := \bigotimes_{n=1}^{N} f_n^{(\ell)}$, for $a(\ell) \neq 0$, and, if $a(\ell) = 0$, set $f(\ell) := 0$.
If $a(\ell) \neq 0$, we have $e(\ell) - f(\ell) = \bigotimes_n e_n^{(\ell_n)} - \bigotimes_n f_n^{(\ell)}$

$$= \sum_{k=1}^{N} \left(\bigotimes_{n=1}^{k-1} f_n^{(\ell)}\right) \otimes \left(e_k^{(\ell_k)} - f_k^{(\ell)}\right) \otimes \left(\bigotimes_{n=k+1}^{N} e_n^{(\ell_n)}\right)$$

Here $\| f_n^{(\ell)} \| \le 1 + \| e_n^{(\ell_n)} \| = 2 \quad \forall n$, and thus

$$\| e(\ell) - f(\ell) \| \le \sum_{k=1}^{N} 2^{k-1} \cdot \varepsilon |a(\ell)| = \frac{1-2^N}{1-2} \varepsilon |a(\ell)|$$

$\le 2^N \varepsilon |a(\ell)|$. Therefore,

$$\sum_{\ell \in I} |a(\ell)| \, \| e(\ell) - f(\ell) \| \le 2^N \varepsilon \sum_{\ell \in I} |a(\ell)|^2 = 2^N \varepsilon \|\psi\|^2 < \infty$$

$\Rightarrow \phi_0 := \sum_{\ell \in I} a(\ell) \left(e(\ell) - f(\ell) \right) \in \mathcal{H}_N \Rightarrow \phi := \phi_0 + \psi \in \mathcal{H}_N$

and $\| \phi - \psi \| \le 2^N \|\psi\|^2 \varepsilon$. Since $\phi = \sum_{\ell \in I} a(\ell) f(\ell)$ and

we can find a sequence $\ell^{(m)} \in I$, $m = 1, 2, \dots,$ such that $\sum_{m=1}^{M} |a(\ell^{(m)})|^2 \xrightarrow{M \to \infty} \sum_{\ell \in I} |a(\ell)|^2$, defining

$\phi^{(M)} := \sum_{m=1}^{M} a(\ell^{(m)}) f(\ell^{(m)})$ shows that $\phi - \phi^{(M)} = \sum_{m=M+1}^{\infty} a(\ell^{(m)}) f(\ell^{(m)})$

$$= \sum_{m > M} a(\ell^{(m)}) \left(f(\ell^{(m)}) - e(\ell^{(m)}) \right) + \sum_{m > M} a(\ell^{(m)}) e(\ell^{(m)})$$

$$\Rightarrow \| \psi - \phi^{(M)} \| \le \| \psi - \phi \| + \| \phi - \phi^{(M)} \|$$
$$\le 2^N \|\psi\|^2 \varepsilon \cdot 2 + \underbrace{\sum_{m > M} |a(\ell^{(m)})|^2}_{\xrightarrow{M \to \infty} 0}$$

$\Rightarrow \exists M_\varepsilon$ s.t. $\| \psi - \phi^{(M_\varepsilon)} \| \le (2^{N+1} \|\psi\|^2 + 1) \varepsilon$.

Since $\phi^{(M_\varepsilon)} \in D(S)$ this shows that $\overline{D(S)} = \mathcal{H}_N$.

This proves that S is a densly defined operator.

For $\psi, \psi' \in D(S)$ we have

$$(\psi', S\psi) = \sum_{k=1}^{M} \sum_{k'=1}^{M'} \lambda_k (\lambda'_{k'})^* \underbrace{\left(\bigotimes_{n=1}^{N} \psi'_{n,k'}, \bigotimes_{n=1}^{N} A_n \psi_{n,k} \right)}$$

$$= \prod_{n=1}^{N} (\psi'_{n,k'}, A_n \psi_{n,k}) = \prod_{n=1}^{N} (A_n \psi'_{n,k'}, \psi_{n,k})$$

$= (S\psi', \psi)$.

$\Rightarrow S$ is symmetric $\overset{5.10.}{\Rightarrow}$ S closable and $\overline{S}$ symmetric.

The proof that $A_S = \overline{S}$ is self-adjoint is given in Reed & Simon I: Theorem VIII.33 and its Corollary. $\square$

b) Typically, the interactions are defined via a potential. In the special case $\mathcal{H}_n = L^2(\mathbb{R}^3)$ $\Rightarrow \mathcal{H}_N = L^2((\mathbb{R}^3)^N)$, these are determined by a function $V_N : (\mathbb{R}^3)^N \to \mathbb{R}$.

The following cases have special names:

* "1-body interaction" = external potential

$$V_N(x) = \sum_{n=1}^{N} V_n(\bar{x}_n), \quad x = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_N) \in (\mathbb{R}^3)^N.$$

* 2-body interaction = pair potential, e.g.,

$$V_N(x) = \sum_{\text{pairs } (n',n)} V_{n'n}(\bar{x}_{n'}, \bar{x}_n).$$

* k-body interaction between particles with labels $1 \le n_1 < n_2 < \dots < n_k \le N$:
$$V_N(x) = \tilde{V}(\bar{x}_{n_1}, \bar{x}_{n_2}, \dots, \bar{x}_{n_k}); \quad \tilde{V} : (\mathbb{R}^3)^k \to \mathbb{R}.$$

* Can also be combined: $V_N(x) = V^{ext} + V^{pair} + \dots$

12.2. Fock spaces

* In both classical and quantum mechanics, the definition of dynamics is typically given for "closed" systems with fixed number of particles.

  What should be done if the number particles inside the system can change?

  Note that this question arises even for systems where the total number of particles is conserved, as soon as we consider dynamics inside a bounded region V of space: particles moving into and away from the region lead to changes in the number of particles in V.

* In quantum mechanics, a natural description of the dynamics is then extending N-particle Hilbert spaces into a Fock space.

12.2.1. Definition (Fock space)

For $N = 1, 2, \dots$, assume that the N-particle dynamics is described by evolution of "wavevectors" in a Hilbert space $\mathcal{H}_N$. The corresponding Fock space is the Hilbert space

$$\mathcal{H}^{(F)} := \bigoplus_{N=0}^{\infty} \mathcal{H}_N$$

where $\mathcal{H}_0 := \mathbb{C}$.

* $\mathcal{H}_0$ is called the vacuum sector, and the vector $\Omega := (1, 0, 0, \dots) \in \mathcal{H}^{(F)}$ is called the vacuum vector.
  ($\mathcal{H}_0$ is a placeholder for 0-particle states.)

* Recall that the probabilistic interpretation of QM requires that wave-vectors ~~are~~ have a unit norm. The same is true for vectors in the Fock space, after we make the following physical "interpretation":

  By def., if $\Psi \in \mathcal{H}^{(F)}$ with $\|\Psi\| = 1$,

  we have $1 = \sum_{N=0}^{\infty} \|\Psi_N\|^2_{\mathcal{H}_N}$. Thus we can

  then identify $p_N := \|\Psi_N\|^2_{\mathcal{H}_N}$ as the probability

  of finding the system with N particles with a wavevector $\frac{1}{\sqrt{p_N}} \Psi_N$ (which is a unit vector in $\mathcal{H}_N$).

* In principle, the spaces $\mathcal{H}_N$ need not to have anything to do with each other. However, the typical N-particle spaces have the following construction:

2.2 Standard constructions for $\mathcal{H}_N$

Suppose the system consists N similar particles, whose 1-particle space is $h =: \mathcal{H}_1$. (For instance, $h = L^2(\mathbb{R}^3)$ (spin-0 particle) or $h = \bigoplus_{\ell=1}^{2s+1} L^2(\mathbb{R}^3)$ (spin-s particle))

* The standard $\mathcal{H}_N$ is then defined as $\bigotimes_{n=1}^{N} h$.

$\Rightarrow$ For spin-0 particles $\mathcal{H}_N \cong L^2(\mathbb{R}^{3N})$.

* If there is no "physical observable" which can distinguish between the particles, the particles are called <u>indistinguishable</u> and it makes a lot of sense to "divide" out the particle-permutation symmetry from the beginning. The following examples are encountered in particle physics

a) Bosons : wavevector is <u>symmetric</u> under permutation of particle labels

b) Fermions : wavevector is <u>antisymmetric</u> ...

Case b) is possible, since only the probability densities $|\psi_N(x)|^2$ are thought to be observable properties, and these remain invariant under multiplications with $e^{i\varphi}$, $\varphi \in \mathbb{R}$, in particular, under $\psi \to -\psi$.

2.3. An aside : Permutation group $S_N :=$
$\{\pi : \{1, 2, \dots, N\} \circlearrowleft \mid \pi \text{ is bijective}\}$

Basic properties : * $|S_N| = N!$

* <u>Transposition</u> (or swap) is a permutation which swaps two elements but leaves others invariant.
* Any $\pi \in S_N$ can be composed from finite number transpositions, and ...

... if the number is even, π has <u>even</u> parity and we define $\operatorname{sgn}(\pi) := +1$.
Otherwise, π has <u>odd parity</u>, and $\operatorname{sgn}(\pi) := -1$.
(These definitions make sense, since the evenness of the number of transposition depends only on π, not on the choice of transpositions used in the decomposition.)

* Commonplace (and convenient) notations:

$$(-1)^\pi := \operatorname{sgn}(\pi) \quad \text{and} \quad (+1)^\pi = +1 \;\forall \pi.$$

<u>2.4. Definition</u>: a) A vector $\Psi \in \bigotimes_{n=1}^{N} h$ is said to be <u>totally symmetric</u>, if $\forall \pi \in S_N$ and $\Phi \in \prod_{n=1}^{N} h$: $\left(\bigotimes_{n=1}^{N} \Phi_{\pi(n)}, \Psi\right) = \left(\bigotimes_{n=1}^{N} \Phi_n, \Psi\right)$

(i.e. if the corresponding multilinear map is totally symmetric)

b) Ψ is <u>totally antisymmetric</u> if $\forall \pi \in S_N$ and $\Phi \in \prod_{n=1}^{N} h$:

$$\left(\bigotimes_{n=1}^{N} \Phi_{\pi(n)}, \Psi\right) = (-1)^\pi \left(\bigotimes_{n=1}^{N} \Phi_n, \Psi\right)$$

($\Rightarrow$ sign-change under swaps)

<u>2.5. Proposition</u>

Denote $\mathcal{H}_N^{(+)} := \{ \Psi \in \bigotimes_{n=1}^{N} h \mid \Psi \text{ is totally symmetric} \}$
$\mathcal{H}_N^{(-)} := \{ \;\text{—''—}\; \mid \;\text{—''—}\; \text{antisymmetric} \}$

Then both $\mathcal{H}_N^{(\sigma)}$, $\sigma = \pm 1$, are closed subspaces and the corresponding orthogonal projections $P_N^{(\sigma)}$ ...

... satisfy for any $\phi \in \prod_{k=1}^{N} h$ and either choice of the sign,

$$P_N^{(\pm)}\Big(\bigotimes_{k=1}^{N} \phi_k\Big) = \frac{1}{N!} \sum_{\pi \in S_N} (\pm 1)^{\pi} \bigotimes_{k=1}^{N} \phi_{\pi(k)}$$

and $P_N^{(\pm)}\Big(\bigotimes_{k=1}^{N} \phi_{\pi(k)}\Big) = (\pm 1)^{\pi} P_N^{(\pm)}\Big(\bigotimes_{k=1}^{N} \phi_k\Big) \quad \forall \pi \in S_N$.

Proof: Exercise ☐

2.6 Definition: a) Bosonic Fock space $= \mathcal{F}^{(+)} := \bigoplus_{N=0}^{\infty} \mathcal{H}_N^{(+)}$
$(\mathcal{H}_0^{(+)} := \mathbb{C})$

b) Fermionic Fock space $= \mathcal{F}^{(-)} := \bigoplus_{N=0}^{\infty} \mathcal{H}_N^{(-)}$
$(\mathcal{H}_0^{(-)} := \mathbb{C})$

* It might look unnecessarily complicated to work with the subspaces $\mathcal{F}^{(\pm)}$ instead of $\mathcal{F}$. However, the restriction to the subspace has some surprising consequences and simplifications. This is particularly so for antisymmetry, which can change the properties of an operator radically. (Stability of matter...)

2.7 An aside: Several species of particles.

If there are K different species of particles (Standard model of particle physics has 24 $s=\frac{1}{2}$ fermions (quarks and leptons & antiparticles) $1+3+8=12$ $s=1$ bosons (gauge bosons) and (usually) a Higgs boson with $s=0$. $\Rightarrow K=37$.), there are K possibly different 1-particle spaces $h^{(k)}$ and the Fock space is

$$\mathcal{F} = \bigoplus_{N \in \mathbb{N}_0^K} \mathcal{H}_N, \quad \text{with } \mathcal{H}_N = \bigotimes_{k=1}^{K} \mathcal{H}_{N_k}^{(k)}; \quad \mathcal{H}_N^{(k)} = P_N^{(\sigma_k)}\Big(\bigotimes_{n=1}^{N} h^{(k)}\Big)$$

where $\sigma_k = -1$ if particle species k is fermionic and $\sigma_k = +1$ if it is bosonic.

* Of course, it is not known if this Fock space is the "right" space for the stand. model ($\exists$ dynamics?)

12.3. "Second quantization"

* In physics, the procedure which produces an operator H_N on the Hilbert space $L^2((\mathbb{R}^3)^N)$ given a function $H(p,q) : (\mathbb{R}^3)^N \times (\mathbb{R}^3)^N \to \mathbb{C}$ is called <u>quantization</u> or <u>first quantization</u>.

* The earlier constructions of the self-adjoint operators from classical Hamiltonian functions $H(p,q) = \sum_{n=1}^{N} \frac{1}{2m_n} \bar{p}_n^2 + V(q)$

  are the prime example of first quantization.

* <u>Second quantization</u> is a procedure

  which takes a "one-particle operator" and produces an operator acting on the corresponding Fock space.:

  Let h denote the one-particle Hilbert space, and $\mathcal{H}_N := \bigotimes_{n=1}^{N} h$, $\mathcal{H}_N^{(\pm)} := P_N^{(\pm)} \mathcal{H}_N$, the corresponding N-particle spaces. For notational convenience, denote $P_N^{(0)} := 1$ and $\mathcal{H}_N^{(0)} := P_N^{(0)} \mathcal{H}_N = \mathcal{H}_N$ for the "distinguishable particle" space.

  Then, if $B \in \mathcal{B}(h)$ is a bounded one-particle operator, we get an N-particle operator $B_N^{(\sigma)}$ of the right symmetry ($\sigma = -1, 0$ or 1) by defining

  $B_N := \overbrace{B \otimes B \otimes \cdots \otimes B}^{N \text{ times}}$ and setting $B_N^{(\sigma)} := P_N^{(\sigma)} B_N \big|_{\mathcal{H}_N^{(\sigma)}}$

  Then $B_N^{(\sigma)} \in \mathcal{B}(\mathcal{H}_N^{(\sigma)})$ with $\|B_N^{(\sigma)}\| \leq \|B_N\| \leq \|B\|^N$, and, if $\psi = P_N^{(\sigma)}(\bigotimes_{n=1}^{N} g_n)$, $g_n \in h$ arbitrary, then for $\sigma = \pm 1$, we have by Prop. 12.2.5. that

$$B_N \psi = \frac{1}{N!} \sum_{\pi \in S_N} (\pm 1)^{\pi} \bigotimes_{n=1}^{N} B g_{\pi(n)} = P_N^{(\sigma)} \Big(\bigotimes_{n=1}^{N} B g_n \Big)$$

$$\Rightarrow B_N^{(\sigma)} \Big(P_N^{(\sigma)} \big(\bigotimes_{n=1}^{N} g_n \big) \Big) = P_N^{(\sigma)} \Big(\bigotimes_{n=1}^{N} B g_n \Big), \text{ for } \sigma = -1, 0, 1. \ (\sigma = 0 \text{ is obvious})$$

This is extended to the full Fock space $\mathcal{F}^{(\sigma)}$ by defining an operator "$\Gamma_\sigma(B)$" by setting $B_0^{(\sigma)} := 1$ and

$$(\Gamma_\sigma(B)\Psi)_N := B_N^{(\sigma)}\Psi_N, \quad N = 0, 1, \ldots$$

for $\Psi \in D(\Gamma_\sigma(B)) := \{ \Psi \in \mathcal{F}^{(\sigma)} \mid \sum_{N=0}^{\infty} \| B_N^{(\sigma)} \Psi_N \|^2 < \infty \}$

The last step of the construction is often denoted by

$$\Gamma_\sigma(B) = \bigoplus_{N=0}^{\infty} B_N^{(\sigma)} .$$

$\Gamma_\sigma(B)$ is called the fermionic / (direct) / bosonic second quantization of B.

* Note that if $\|B\| \leq 1$, then $D(\Gamma_\sigma(B)) = \mathcal{F}^{(\sigma)}$ and $\|\Gamma_\sigma(B)\| \leq 1$.

* If $U \in \mathcal{B}(h)$ is unitary, then $\Gamma_\sigma(U) \in \mathcal{B}(\mathcal{F}^{(\sigma)})$ is also unitary, and $\Gamma_\sigma(U)^* = \Gamma_\sigma(U^*)$.
  (Show first that $(U_N)^* = (U^*)_N$ by looking at arbitrary $(\otimes g_n, (U^*)_N (\otimes g_n'))$. Check similarly that $(U^*)_N U_N = 1 = U_N (U^*)_N$.)

* If $U_t = e^{-itH}$ is a strongly contin. unit. semigroup on h, with a generator H, then $t \mapsto \Gamma_\sigma(U_t)$ is a strongly contin. unitary semigroup on $\mathcal{F}^{(\sigma)}$ whose generator is denoted by "$d\Gamma_\sigma(H)$". ($\Rightarrow \Gamma_\sigma(U_t) = e^{-it\, d\Gamma_\sigma(H)}$).

  $d\Gamma_\sigma(H)$ can be constructed from H similarly to above case: Define $S_N^{(\sigma)}$ on $D(S_N^{(\sigma)}) := P_N^{(\sigma)}(\operatorname{span}\{\otimes g_n \mid g_n \in D(H)\})$ by $S_N^{(\sigma)}(P_N^{(\sigma)}(\otimes g_n)) = P_N^{(\sigma)}(\otimes H g_n)$, set $D(S^{(\sigma)}) = \{\Psi \in \mathcal{F}^{(\sigma)} \mid \exists N_0 \text{ s.t. } \Psi_N = 0 \ \forall N \geq N_0\}$ and define $(S^{(\sigma)}\Psi)_N := S_N^{(\sigma)}\Psi_N \ \forall N, \Psi \in D(S^{(\sigma)})$.
  Then $S^{(\sigma)}$ is symmetric $\Rightarrow$ closable, and

$$d\Gamma_\sigma(H) = \overline{S^{(\sigma)}} .$$

* Example: $t \mapsto U_t := e^{-it}1$ generates a strongly contin. USG on any Hilbert space h. Its generator is $H = 1$. The corresponding second quantization is denoted by $\hat{N} := d\Gamma_\sigma(1)$, and it is called the number operator.

Explicitly, then $(\hat{N}\Psi)_N = N\Psi_N$, $N = 0, 1, \dots$, with $D(\hat{N}) = \{\Psi \in \mathcal{F}^{(0)} \mid \sum_{N=0}^{\infty} N^2 \|\Psi_N\|^2 < \infty\}$.

12.4. Creation and annihilation operators

In the "direct" Fock space $\mathcal{F}^{(0)} := \bigoplus_{N=0}^{\infty} \mathcal{H}_N$, $\mathcal{H}_N := \bigotimes_{n=1}^{N} h$, one can define operators $c(g)$ and $a(g)$, $g \in h$, such that $c(g)$ "creates a particle with label 1 at a state g" and $a(g)$ "annihilates the particle 1, projected to state g". The mathematical definition of $c(g)$ and $a(g)$ are explained in detail in Exercise 13.3. They are defined via a sequence of contin. (i.e. bounded) linear maps $c_N : \mathcal{H}_N \to \mathcal{H}_{N+1}$, $N = 0, 1, \dots$, and $a_N : \mathcal{H}_N \to \mathcal{H}_{N-1}$, $N = 1, 2, \dots$; which satisfy

$$c_N\Big(\bigotimes_{n=1}^{N} \psi_n\Big) = \sqrt{N+1}\, g \otimes \Big(\bigotimes_{n=1}^{N} \psi_n\Big), \quad \text{and}$$

$$a_N\Big(\bigotimes_{n=1}^{N} \psi_n\Big) = \sqrt{N}\, \langle g|\psi_1\rangle_h \bigotimes_{n=2}^{N} \psi_n .$$

Then one sets $(a(g)\Psi)_N = a_{N+1}\Psi_{N+1}$ $(\in \mathcal{H}_N)$

and $(c(g)\Psi)_0 = 0$, $(c(g)\Psi)_N = c_{N-1}\Psi_{N-1}$, $N = 1, 2, \dots$ on the domain $D(\sqrt{\hat{N}}) := \{\Psi \mid \sum_N N\|\Psi_N\|^2 < \infty\} =: D_0$. Note that order is here important, and the labels of the other particles get shifted as one operates with c_N and a_N. Note: $g \mapsto c(g)$ is linear, but $g \mapsto a(g)$ conj.lin.

However, these operators are seldom used alone, but instead one considers their bosonic and fermionic projections by $P^{(\pm)}$ to $\mathcal{F}^{(\pm)}$: Set $D_\sigma := D_0 \cap \mathcal{F}^{(\sigma)}$,

$$c_\sigma(g) := \overline{P^{(\sigma)} c(g)|_{D_\sigma}}, \quad a_\sigma(g) := \overline{P^{(\sigma)} a(g)|_{D_\sigma}}, \quad \sigma = \pm 1.$$

12.4.1 Fermionic creation and annihilation operators

Perhaps surprisingly, it is the fermionic operators, which work with antisymmetric functions, that have better regularity properties. In particular,

4.1.1. Theorem: $\forall g \in h$ we have $a_-(g), c_-(g) \in \mathcal{B}(\mathcal{F}^{(-)})$ with $\|a_-(g)\| = \|g\|_h = \|c_-(g)\|$.
In addition, $c_-(g) = (a_-(g))^*$.

Proof: Exercise 13,4. □

Consequently, one writes $a_-^*(g)$ instead of $c_-(g)$. Note that the somewhat arbitrary looking scale factors $\mathcal{O}(N^{\frac{1}{2}})$ in the definition of a_N and c_N are chosen so that after the particle permutations involved in $P^{(\pm)}$ one gets the correct scaling, such as $\|c_-(g)\| = \|g\|$ above. In particular, then

4.1.2. Proposition: If $N \in \mathbb{N}_+$, $g \in h^N$ are given, define $\Psi \in \mathcal{F}^{(0)}$ by setting $\Psi_N := \bigotimes_{n=1}^{N} g_n \in \mathcal{H}_N$ and $\Psi_{N'} := 0$ for $N' \neq N$.
Then $P^{(-)}\Psi = \frac{1}{\sqrt{N!}} a_-^*(g_1) \cdots a_-^*(g_N)\Omega$, Ω = vacuum vector.

4.1.3. Theorem: Suppose $(e_\ell)_{\ell \in I}$ form an ONB for h, and let $\Omega := (1, 0, 0, ..) \in \mathcal{F}^{(-)}$ denote the vacuum vector. Let $I^{(-)}$ be a collection of sequences $\ell = (\ell_1, \ldots, \ell_N)$, $0 \leq N < \infty$, such that

a) $\ell_j \neq \ell_{j'}$ $\forall j' \neq j$ (non-repeating)
b) If $\ell \in I^N$ is non-repeating, $\exists$ unique $\pi \in S_N$ and $\ell' \in I^{(-)}$ such that $\ell_n = \ell'_{\pi(n)}$ $\forall n = 1, \ldots, N$, and ℓ' has length N.

Then the collection of vectors $e(\ell) := a_-^*(e_{\ell_1}) \cdots a_-^*(e_{\ell_N})\Omega \in \mathcal{F}^{(-)}$; $\ell \in I^{(-)}$ & setting $e(\emptyset) := \Omega$, forms an ONB of $\mathcal{F}^{(-)}$.

4.1.4. Theorem: For every $f, g \in h$, the following "canonical anticommutation relations" hold: denoting $\{A, B\} := AB + BA$,

$$\{a_-(f), a_-(g)\} = 0 = \{a_-^*(f), a_-^*(g)\},$$

and $$\{a_-(f), a_-^*(g)\} = \langle f | g \rangle_h \, 1 .$$

In particular, $a_-(f)^2 = 0 = a_-^*(f)^2$.

Proof of 4.1.2.: Induction in N:

If $N = 1$, then $\Psi = (0, g, 0, 0, \dots)$ and since $\Omega \in D_- := D_0 \cap \mathcal{F}^{(-)}$, $a_-^*(g)\Omega = P^{(-)} c(g)\Omega = P^{(-)}(0, g, 0, \dots) = P^{(-)}\Psi$. $\therefore$ Claim holds.

Assume $N \geq 2$, and suppose claim true for sets of size $< N$. Then $a_-^*(g_2) \cdots a_-^*(g_N)\Omega$

$$= \sqrt{(N-1)!}\, P^{(-)}(0, \dots, \underset{\uparrow_{N-1}}{\bigotimes_{n=2}^{N} g_n}, 0, \dots)$$

$$= \frac{1}{\sqrt{(N-1)!}} \sum_{\pi' \in S' := S_{N-1}(\{2, \dots, N\})} (-1)^{\pi'} (0, \dots, \underset{\uparrow_{N-1}}{\bigotimes_{n=2}^{N} g_{\pi'(n)}}, 0, \dots) \in D_-$$

Since $c(g_1)(0, \dots, \underset{\uparrow_{N-1}}{\bigotimes_{n=2}^{N} g_{\pi'(n)}}, 0, \dots)$

$$= (0, \dots, 0, \underset{\uparrow_N}{\sqrt{N}\, g_1 \otimes (\bigotimes_{n=2}^{N} g_{\pi'(n)})}, 0, \dots)$$

$$\Rightarrow a_-^*(g_1) \cdots a_-^*(g_N)\Omega = \frac{1}{\sqrt{(N-1)!}} \sum_{\pi' \in S'} (-1)^{\pi'} P^{(-)}(0, \dots, \underset{\uparrow_N}{\sqrt{N}\, g_1 \otimes (\bigotimes_{n=2}^{N} g_{\pi'(n)})}, 0, \dots)$$

$= (0, \dots, \Phi_N, 0, \dots)$ where, denoting $\tilde{\pi}(1) = 1$, $\tilde{\pi}(n) = \pi'(n)$, $n > 1$, $\Rightarrow \tilde{\pi} \in S_N$.

$$\Phi_N = \frac{1}{\sqrt{(N-1)!}} \sum_{\pi' \in S'} (-1)^{\pi'} \frac{\sqrt{N}}{N!} \sum_{\pi \in S_N} (-1)^{\pi} \bigotimes_{n=1}^{N} g_{\pi(\tilde{\pi}(n))}$$

Since S_N is a group, can sum over $\pi_0 := \pi \circ \tilde{\pi}$ instead of π, i.e. $\pi = \pi_0 \circ \tilde{\pi}^{-1}$ and $\sum_{\pi \in S_N} \cdots = \sum_{\pi_0 \in S_N} \cdots \Big|_{\pi = \pi_0 \circ \tilde{\pi}^{-1}}$.

Here $(-1)^{\pi} = (-1)^{\pi_0} (-1)^{\tilde{\pi}^{-1}} = (-1)^{\pi_0} (-1)^{\tilde{\pi}} = (-1)^{\pi_0} (-1)^{\pi'}$, since if $\pi' = s_1 \circ \cdots \circ s_M$ represent π' in terms of swaps s_i,

then $\tilde{\pi} = \tilde{S}_1 \circ \dots \circ \tilde{S}_M$ ($\tilde{S}_i(j) = S_i(j)$, and use $\tilde{\pi}(1) = 1$)
and $\tilde{\pi}^{-1} = \tilde{S}_M \circ \dots \circ \tilde{S}_1$. Therefore,

$$\Phi_N = \frac{1}{\sqrt{N!}} \frac{1}{(N-1)!} \sum_{\pi_0 \in S_N} (-1)^{\pi_0} \Big(\underbrace{\sum_{\pi' \in S'} 1}_{= (N-1)!} \Big) \bigotimes_{n=1}^{N} g_{\pi_0(n)}$$

$$= \sqrt{N!}\, \underline{P}_N^{(-)} \Big(\bigotimes_{n=1}^{N} g_n \Big)$$

Hence, $a_-^*(g_1) \cdots a_-^*(g_N)\Omega = \sqrt{N!}\, \underline{P}^{(-)} \Psi$.
This completes the induction step. □

Changing "-1" $\to$ "$+1$" in the above proof also yields the following result for bosons:

<u>4.1.5. Lemma</u>: If $N \in \mathbb{N}_+$, $g \in h^N$ are given and we define $\Psi_N := \bigotimes_{n=1}^{N} g_n$ and $\Psi_n = 0$, for $n \neq N$, then

$$\underline{P}^{(+)} \Psi = \frac{1}{\sqrt{N!}} \tilde{c}_+(g_1) \cdots \tilde{c}_+(g_N)\Omega$$

where $\tilde{c}_+(g) := \underline{P}^{(+)} c(g)|_{D_+}$ is a densely defined operator on $\mathcal{F}^{(+)}$ and $D_+ := D_0 \cap \mathcal{F}^{(+)}$.

<u>Proof of 4.1.3.</u>: Define $I^{(-)}$ and $e(\ell)$ as in the theorem. Consider then some $\ell, \ell' \in I^{(-)}$ and denote $N = |\ell|$, $N' = |\ell'|$ and $\Psi = (0, \dots, 0, \bigotimes_{n=1}^{N} e_{\ell_n}, 0, \dots)$, $\Psi' = (0, \dots, 0, \bigotimes_{n=1}^{N'} e_{\ell'_n}, 0, \dots)$, with the nonzero entries at positions N and N'.

By Proposition 4.1.2., then
$\langle e(\ell) | e(\ell') \rangle = \sqrt{N! \cdot N'!}\, \langle \underline{P}^{(-)} \Psi \,|\, \underline{P}^{(-)} \Psi' \rangle$
Thus if $N' \neq N \Rightarrow \langle e(\ell) | e(\ell') \rangle = 0$. If $N' = N$, then
$\langle e(\ell) | e(\ell') \rangle = N! \, \langle \underline{P}_N^{(-)} (\bigotimes_{n=1}^{N} e_{\ell_n}) \,|\, \underline{P}_N^{(-)} (\bigotimes_{n=1}^{N} e_{\ell'_n}) \rangle$

$$= N! \, \Big\langle \bigotimes_{n=1}^{N} e_{\ell_n} \,\Big|\, \underline{P}_N^{(-)} \Big(\bigotimes_{n=1}^{N} e_{\ell'_n}\Big) \Big\rangle$$

$\underline{P}_N^{(-)}$ self-adjoint proj.

$$= \sum_{\pi \in S_N} (-1)^{\pi} \underbrace{\prod_{n=1}^{N} \langle e_{\ell_n} | e_{\ell'_{\pi(n)}} \rangle}$$

$= 0$, unless $\ell_n = \ell'_{\pi(n)}$ $\forall n$.
$\Rightarrow \ell = \ell'$ and $\pi = \mathrm{id}$, by def. of $I^{(-)}$.

Hence, $\langle e(\ell) | e(\ell') \rangle = \mathbb{1}(\ell' = \ell)$ and the set $(e(\ell))_{\ell \in I^{(-)}}$ is orthonormal. If $\Phi, \Psi \in \mathcal{F}^{(-)}$,

then $\langle \Phi | \Psi \rangle_{\mathcal{F}^{(-)}} = \langle \Phi | \Psi \rangle_{\mathcal{F}^{(\omega)}} = \sum_{N=0}^{\infty} \langle \Phi_N | \Psi_N \rangle_{\mathcal{H}_N}$.

Here, $\langle \Phi_N | \Psi_N \rangle_{\mathcal{H}_N} = \sum_{\ell \in I^N} \langle \Phi_N | \tilde{e}(\ell) \rangle \langle \tilde{e}(\ell) | \Psi_N \rangle$

since $\tilde{e}(\ell) := \bigotimes_{n=1}^{N} e_{\ell_n}$ is an ONB for $\mathcal{H}_N$. As

$\Psi_N \in \mathcal{H}_N^{(-)} \Rightarrow P_N^{(-)} \Psi_N = \Psi_N$ and thus $\langle \tilde{e}(\ell) | \Psi_N \rangle$

$= \langle P_N^{(-)} \tilde{e}(\ell) | \Psi_N \rangle$.

Suppose first that $\ell \in I^N$ has a repeated index, i.e., that $\ell_{j'} = \ell_j$ for some $j' \neq j$. Denote the swap $j' \leftrightarrow j$ by $\pi \in S_N$, and define $\ell'_n := \ell_{\pi(n)}$. Since π swaps identical elements, then $\ell' = \ell$, and thus $\tilde{e}(\ell') = \tilde{e}(\ell)$. However, by Prop. 2.5., we have

$P_N^{(-)}(\tilde{e}(\ell')) = P_N^{(-)}(\bigotimes_{n=1}^{N} e_{\ell_{\pi(n)}}) = (-1)^{\pi} P_N^{(-)}(\bigotimes_{n=1}^{N} e_{\ell_n})$

$= - P_N^{(-)}(\tilde{e}(\ell))$

$\Rightarrow P_N^{(-)}(\tilde{e}(\ell)) = - P_N^{(-)}(\tilde{e}(\ell)) \Rightarrow P_N^{(-)}(\tilde{e}(\ell)) = 0$.

$\Rightarrow \langle \tilde{e}(\ell) | \Psi_N \rangle = 0$.

Then we are left with $\ell \in I^N$ which have no repetitions. By assumption, there is then exactly one $\pi \in S_N$ and $\ell' \in I^{(-)}$ such that $\ell_n = \ell'_{\pi(n)}$. By Prop. 2.5, then $P_N^{(-)}(\tilde{e}(\ell)) = P_N^{(-)}(\bigotimes_{n=1}^{N} e_{\ell'_{\pi(n)}}) = (-1)^{\pi} P_N^{(-)}(\tilde{e}(\ell'))$.

$\Rightarrow \langle \Phi_N | \tilde{e}(\ell) \rangle \langle \tilde{e}(\ell) | \Psi_N \rangle = \langle \Phi_N | P_N^{(-)} \tilde{e}(\ell) \rangle \langle P_N^{(-)} \tilde{e}(\ell) | \Psi_N \rangle$

$= \underbrace{(-1)^{\pi} \cdot (-1)^{\pi}}_{=+1} \langle \Phi_N | P_N^{(-)} \tilde{e}(\ell') \rangle \langle P_N^{(-)} \tilde{e}(\ell') | \Psi_N \rangle$

$\overset{4.1.2.}{=} \frac{1}{N!} \langle \Phi_N | e(\ell') \rangle \langle e(\ell') | \Psi \rangle$

Since to every $\ell' \in I^{(-)}$ with N elements there are exactly $N!$ non-repeating sequences $\ell \in I^N$ obtained from permutations of ℓ', we can conclude that

$\langle \Phi | \Psi \rangle_{\mathcal{F}^{(-)}} = \sum_{N=0}^{\infty} \sum_{\ell \in I^N, \text{ non-repeating}} \langle \Phi_N | \tilde{e}(\ell) \rangle \langle \tilde{e}(\ell) | \Psi_N \rangle$

$= \sum_{\ell' \in I^{(-)}} \langle \Phi | e(\ell') \rangle \langle e(\ell') | \Psi \rangle$. Thus $(\tilde{e}(\ell'))_{\ell' \in I^{(-)}}$ is ONB $\square$

Proof of 4.14: Suppose $\ell \in I^{(-)}$, $|\ell| = N$, and consider $e(\ell)$ and $\tilde{e}(\ell)$ as above. By Prop. 4.1.2., for any $f, g \in h$,

$a_-^*(f)a_-^*(g)e(\ell) = \sqrt{(N+2)!}\, P^{(-)}(0,\dots, f\otimes g\otimes \tilde{e}(\ell), 0, \dots)$
$= -\sqrt{(N+2)!}\, P^{(-)}(0,\dots, g\otimes f\otimes \tilde{e}(\ell), 0,\dots) = -a_-^*(g)a_-^*(f)e(\ell)$.
Thus if $\Phi, \Psi \in \mathcal{F}^{(-)}$:

$$\langle \Phi | a_-^*(f)a_-^*(g)\Psi\rangle = \sum_{\ell\in I^{(-)}} \langle \Phi | a_-^*(f)a_-^*(g)e(\ell)\rangle\langle e(\ell)|\Psi\rangle$$

$$= -\langle \Phi | a_-^*(g)a_-^*(f)\Psi\rangle \Rightarrow a_-^*(f)a_-^*(g) = -a_-^*(g)a_-^*(f).$$

Since $a_-^*(f) = (a_-(f))^*$, taking an adjoint yields also $a_-(f)a_-(g) = (a_-^*(g)a_-^*(f))^* = -a_-(g)a_-(f)$. Thus, if $f=g$, we have in particular $a_-(f)^2 = 0 = a_-^*(f)^2$.

For the missing equation, fix $f, g \in h$. Then $M_0 := \operatorname{span}\{f,g\}$ is a closed subspace, $\dim M_0 \le 2$. If $f=0$ or $g=0 \Rightarrow a_-(f)=0$ or $a_-^*(g)=0$ (thrm. 4.1.1.)
$\Rightarrow a_-(f)a_-^*(g) + a_-^*(g)a_-(f) = 0 = \langle f|g\rangle 1$.
Thus can assume $f,g\neq 0$. We choose an ONB for M_0 s.t. $e_0 := \frac{1}{\|f\|}f$. Then $g = \langle e_0|g\rangle e_0 + \langle e_1|g\rangle e_1$ where $e_1 = 0$ if $\dim M_0 = 1$, and $\|e_1\|=1$ if $\dim M_0 = 2$. By choosing some ONB for $M_0^\perp$ we obtain an ONB $(e_\ell)_{\ell\in I}$ for h. Then we construct $I^{(-)}$ by requiring that any $\ell\in I^{(-)}$ is ordered so that e_0 comes first, followed by e_1, whenever either or both are present. Let $e(\ell)$, $\ell\in I^{(-)}$, denote corresponding ONB for $\mathcal{F}^{(-)}$.

Recall Prop. 4.1.2. If $\ell\in I^{(-)}$ with $|\ell| = N$, then for $N=0$, we have $a_-^*(g)a_-(f)e(\ell) = 0$, and for $N\ge 1$:

$$a_-(f)e(\ell) = \frac{1}{\sqrt{N!}}\sum_{\pi\in S_N}(-1)^\pi \sqrt{N}\underbrace{\langle f|e_{\ell_{\pi(1)}}\rangle}_{=0 \text{ if } \pi(1)\neq 0}(0,\dots,\underset{\uparrow\, N-1}{\bigotimes_{n=2}^{N}} e_{\ell_{\pi(n)}}, 0,\dots)$$

$$= \frac{1}{\sqrt{(N-1)!}}\sum_{\pi\in S_N}(-1)^\pi \mathbb{1}(\ell_{\pi(1)}=0)\cdot\|f\|\,(0,\dots,\bigotimes_{n=2}^{N} e_{\ell_{\pi(n)}}, 0,\dots)$$

This equals 0 if $0\notin\ell$, and if $0\in\ell$, then by construction $0=\ell_1$. Hence, $\mathbb{1}(\ell_{\pi(1)}=0) = \mathbb{1}(0\in\ell)\,\mathbb{1}(\pi(1)=1)$. Now $\pi\in S_N$ with $\pi(1)=1$ are in 1-1 correspondence with $\pi'\in S_{N-1}'(\{2,\dots,N\}) =: S_{N-1}'$ and $(-1)^\pi = (-1)^{\pi'}$. Hence

$$a_-(f)e(\ell) = \mathbb{1}(0\in\ell)\|f\|\frac{1}{\sqrt{(N-1)!}}\sum_{\pi'\in S_{N-1}'}(-1)^{\pi'}(0,\dots,\bigotimes_{n=2}^{N} e_{\ell_{\pi'(n)}}, 0,\dots)$$

$\overset{4.1.2.}{=} \mathbb{1}(0\in\ell)\|f\|\, a_-^*(e_{\ell_2})\cdots a_-^*(e_{\ell_N})\Omega$. By linearity of $g\mapsto a_-^*(g)$, $\forall \ell$, including $|\ell|=0$, we have

$$a_-^*(g)a_-(f)e(\ell) = \mathbb{1}(0\in\ell)\|f\|\langle \tfrac{f}{\|f\|}|g\rangle \underbrace{a_-^*(e_{\ell_1})\cdots a_-^*(e_{\ell_N})\Omega}_{=e(\ell)}$$
$$+ \mathbb{1}(0\in\ell)\|f\|\langle e_1|g\rangle a_-^*(e_1)a_-^*(e_{\ell_2})\cdots a_-^*(e_{\ell_N})\Omega$$

On the other hand, $\forall \ell \in I^{(-)}$,

$$a_-(f)a_-^*(g)e(\ell) = a_-(f)\Big[\langle e_0|g\rangle a_-^*(e_0)e(\ell)$$
$$+ \langle e_1|g\rangle a_-^*(e_1)e(\ell)\Big]_N$$
$$= \langle e_0|g\rangle\, a_-(f)\, \underline{P}^{(-)}\Big(0,\dots, \bigotimes_{n=0}^{N} e_{\ell_n}, 0, \dots\Big)\Big|_{\ell_0=0} \cdot \sqrt{(N+1)!}$$
$$+ \langle e_1|g\rangle\, a_-(f)\, \underline{P}^{(-)}\Big(0,\dots, \bigotimes_{n=0}^{N} e_{\ell_n}, 0, \dots\Big)\Big|_{\ell_0=1} \cdot \sqrt{(N+1)!}$$

Here $\sqrt{(N+1)!}\, a_-(f)\underline{P}^{(-)}\Big(0,\dots, \bigotimes_{n=0}^{N} e_{\ell_n}, 0, \dots\Big) = \Big(0,\dots, \underset{\uparrow N}{\Phi_N}, 0, \dots\Big)$

with $\Phi_N = \frac{1}{\sqrt{(N+1)!}} \sum_{\pi \in S_{N+1}(\{0,1,\dots,N\})} (-1)^\pi \sqrt{N+1}\, \langle f | e_{\ell_{\pi(0)}}\rangle \bigotimes_{n=1}^{N} e_{\ell_{\pi(n)}}$
$= \|f\|\, \mathbb{1}(0 \in \ell)\, \frac{1}{\sqrt{N!}} \sum_{\pi \in S_{N+1}} (-1)^\pi\, \mathbb{1}(\ell_{\pi(0)} = 0) \bigotimes_{n=1}^{N} e_{\ell_{\pi(n)}}$

By earlier results, if $\ell \in I^{(-)}$ is such that $0 \in \ell$, then $a_-^*(e_0)^2 = 0$ and by anticommutations of all a_-^* we have $a_-^*(e_0)e(\ell) = 0$. If $0 \notin \ell$, then
$\Phi_N\big|_{\ell_0=0} = \|f\| \frac{1}{\sqrt{N!}} \sum_{\pi \in S_{N+1}} (-1)^\pi\, \mathbb{1}(\pi(0)=0) \bigotimes_{n=1}^{N} e_{\ell_{\pi(n)}}$
$= \|f\| \sqrt{N!}\, \underline{P}_N^{(-)}\Big(\bigotimes_{n=1}^{N} e_{\ell_n}\Big)$. Thus $\langle e_0|g\rangle a_-(f)a_-^*(e_0)e(\ell)$
$= \|f\| \langle \frac{f}{\|f\|} | g\rangle\, \mathbb{1}(0 \notin \ell)\, e(\ell) \quad = \mathbb{1}(0 \notin \ell)\langle f|g\rangle e(\ell)$.

For the second term, if $e_1 = 0$ or $0 \notin \ell$, it is zero.
If $e_1 \neq 0$ and $0 \in \ell$, it is zero if $1 \in \ell$, and else
$\Phi_N = \|f\| \frac{1}{\sqrt{N!}} \sum_{\pi \in S_{N+1}} (-1)^\pi\, \mathbb{1}(\pi(0) = 1) \bigotimes_{n=1}^{N} e_{\ell_{\pi(n)}}$
$\pi_0 = s_{01} \circ \pi$
$\overset{\uparrow}{=} -\|f\| \frac{1}{\sqrt{N!}} \sum_{\pi_0 \in S_{N+1}} (-1)^{\pi_0}\, \mathbb{1}(\pi_0(0) = 0) \bigotimes_{n=1}^{N} e_{\tilde{\ell}_{\pi_0(n)}} \Big|_{\substack{\tilde{\ell}_1 = 1 \\ \tilde{\ell}_n = \ell_n,\, n \ge 2.}}$

(Note that by construction $0 \in \ell \Rightarrow \ell_1 = 0$, hence the original order is $(1, 0, \ell_2, \dots) \xrightarrow{s_{01}} (0, 1, \ell_2, \dots)$.)
Therefore, $a_-(f)a_-^*(e_1)e(\ell) = \mathbb{1}(e_1 \neq 0, 0 \in \ell, 1 \notin \ell)$
$\times (-\|f\|)\, a_-^*(e_1)a_-^*(e_{\ell_2}) \cdots a_-^*(e_{\ell_N})\Omega$.
$= -\mathbb{1}(0 \in \ell)\, \|f\|\, a_-^*(e_1)a_-^*(e_{\ell_2}) \cdots a_-^*(e_{\ell_N})\Omega$. $\quad (0, 1 \in \ell \Rightarrow \ell_2 = 1 \Rightarrow \text{zero})$

Thus, collecting the results together, $\forall \ell \in I^{(-)}$:
$(a_-^*(g)a_-(f) + a_-(f)a_-^*(g))e(\ell)$
$= (f, g)\,[\mathbb{1}(0 \in \ell) + \mathbb{1}(0 \notin \ell)]\, e(\ell) + 0 = \langle f|g\rangle e(\ell)$.
Since $e(\ell)$ forms an ONB,
$\Rightarrow a_-^*(g)a_-(f) + a_-(f)a_-^*(g) = \langle f|g\rangle \mathbb{1} \quad \square$