9. Self-adjoint extensions of symmetric operators

* In this section we assume that S is a densely defined, symmetric operator

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9.1. Assumption: S is an operator on $\mathcal{H}$ for which
a) $\overline{D(S)} = \mathcal{H}$
b) $\forall \phi, \psi \in D(S): \quad (\phi, S\psi) = (S\phi, \psi)$

* If S satisfies a) and b), then by Thrm 5.10.c) $\Rightarrow$ S is closable, and $\bar{S}$ satisfies Assumption 9.1.
* If $V \in C^\infty(\mathbb{R}^d, \mathbb{R})$, then $S = (H_0 + V)|_D$ is symmetric with the domain $D := C_c^\infty(\mathbb{R}^d)$.

(Proof: $\psi, \phi \in D \Rightarrow (\phi, S\psi) = \underbrace{(\phi, -\tfrac{1}{2}\Delta\psi)}_{\overset{\text{part.int.}}{=} (-\frac{1}{2}\Delta\phi, \psi)} + \underbrace{(\phi, V\psi)}_{\overset{V \text{ is real}}{=} (V\phi, \psi)} = (S\phi, \psi)_\square$)

* The goal of this section is to find out if such operators S have self-adjoint extensions, and what these might be. This can be done via the following tool:

9.2. Proposition: Let T be a symmetric operator: $\forall \phi, \psi \in D(T)$, $(\phi, T\psi) = (T\phi, \psi)$.

Then there exists a unique mapping $C_T : R(T+i) \to R(T-i)$ for which
$$C_T(T\psi + i\psi) = T\psi - i\psi \quad \forall \psi \in D(T).$$
In addition, C_T is an operator, bijective, and an isometry.

Proof: Let $R_\pm := R(T \pm i) := \{T\psi \pm i\psi \mid \psi \in D(T)\}$.
Clearly, R_+ and R_- are subspaces of $\mathcal{H}$.
Also, for any $\psi \in D(T)$,

$$\|T\psi \pm i\psi\|^2 = (T\psi, T\psi) + (T\psi, \pm i\psi) + (\pm i\psi, T\psi) + (\pm i\psi, \pm i\psi)$$
$$= \|T\psi\|^2 + \|\psi\|^2 \pm i \underbrace{(T\psi, \psi)}_{= (\psi, T\psi)} \mp i(\psi, T\psi)$$

Therefore, as T is symmetric,

(*) $$\|T\psi \pm i\psi\|^2 = \|T\psi\|^2 + \|\psi\|^2 \quad \forall \psi \in D(T).$$

Thus, if $(T+i)\psi = 0 \Rightarrow 0 = \|T\psi\|^2 + \|\psi\|^2$ $\Rightarrow \psi = 0$. Therefore, $T+i$ is one-to-one, and there is a mapping $(T+i)^{-1} : R(T+i) \to D(T+i)$. Since $D(T+i) = D(T) = D(T-i)$, we can then define $\mathcal{C}_T := (T-i)(T+i)^{-1}$ for which $D(\mathcal{C}_T) = R(T+i)$ and $R(\mathcal{C}_T) = R(T-i)$. But as also $T-i$ is one-to-one by (*), $\mathcal{C}_T$ is invertible, and $\mathcal{C}_T^{-1} = (T+i)(T-i)^{-1}$.
Clearly, then $\forall \psi \in D(T) : \mathcal{C}_T(T\psi + i\psi)$
$$\underset{D(T+i)=D(T)}{=} \mathcal{C}_T(T+i)\psi = (T-i)\psi = T\psi - i\psi.$$
If $\mathcal{C}_T'$ is another mapping $R(T+i) \to R(T-i)$ with this property, then $\forall \phi \in R(T+i)$
$$\mathcal{C}_T'\phi = \mathcal{C}_T'(T+i)\underbrace{(T+i)^{-1}\phi}_{=:\psi} = \mathcal{C}_T'(T\psi + i\psi)$$
$$= T\psi - i\psi = (T-i)(T+i)^{-1}\phi = \mathcal{C}_T\phi$$
$\Rightarrow \mathcal{C}_T' = \mathcal{C}_T$. This proves the stated uniqueness.
$\mathcal{C}_T$ is bijective, and it is linear, since the inverse of an operator is always linear.
(A operator, one-to-one $\Rightarrow \exists A^{-1} : R(A) \to D(A)$, and $\forall \psi_1, \psi_2 \in R(A)$, $\alpha_1, \alpha_2 \in \mathbb{C}$, let $\phi_1 = A^{-1}\psi_1$, $\phi_2 = A^{-1}\psi_2$ when $\alpha_1\psi_1 + \alpha_2\psi_2 \in R(A)$ and $\alpha_1\psi_1 + \alpha_2\psi_2 = \alpha_1 A\phi_1 + \alpha_2 A\phi_2$
$$= A(\alpha_1\phi_1 + \alpha_2\phi_2)$$
$$\Rightarrow A^{-1}(\alpha_1\psi_1 + \alpha_2\psi_2) = \alpha_1\phi_1 + \alpha_2\phi_2 = \alpha_1 A^{-1}\psi_1 + \alpha_2 A^{-1}\psi_2.)$$
Thus $\mathcal{C}_T$ is an operator. Now, if $\psi \in R(T+i)$ $\Rightarrow \exists \phi = (T+i)^{-1}\psi \in D(T)$, and

$$\|\mathcal{C}_T\psi\|^2 = \|(T-i)\phi\|^2 \overset{(*)}{=} \|T\phi\|^2 + \|\phi\|^2$$
$$\underset{(*)}{=} \|(T+i)\phi\|^2 = \|\psi\|^2.$$

$\Rightarrow \|\mathcal{C}_T\psi\| = \|\psi\| \;\; \forall \psi \in D(\mathcal{C}_T)$, and $\mathcal{C}_T$ is an isometry. $\square$

9.3. Definition For a symmetric operator T on $\mathcal{H}$, the linear isometry $\mathcal{C}_T$ defined in Prop. 9.2. is called the Cayley-transform of T.

9.4. Theorem Let T and T' be symmetric operators. Then

o) $R(1-\mathcal{C}_T) = D(T)$ and $1-\mathcal{C}_T$ is 1-1.

a) $\mathcal{C}_{T'} = \mathcal{C}_T \Rightarrow T' = T$

b) $T \subset T' \Leftrightarrow \mathcal{C}_T \subset \mathcal{C}_{T'}$

c) T is closed $\Leftrightarrow \mathcal{C}_T$ is closed

d) If T is densely defined, T is closable and

$$\mathcal{C}_{\bar{T}} = \overline{\mathcal{C}_T}.$$

Conversely, if V is an operator on $\mathcal{H}$ for which $\|V\psi\| = \|\psi\| \;\forall \psi \in D(V)$ ($\Leftrightarrow V$ is an isometry) and $1-V$ is one-to-one, then the mapping

$$T = i(1+V)(1-V)^{-1} : R(1-V) \to R(1+V)$$

is a symmetric operator, and $\mathcal{C}_T = V$.

* In summary: Taking a closure or finding symmetric extensions of a symmetric operator can be done by closure or isometric extensions of its Cayley-transform.

Proof: Let us first prove "o)" and then the "converse". Suppose $\psi \in R(1-\mathcal{C}_T)$
$\Rightarrow \exists \phi \in D(\mathcal{C}_T) = R(T+i)$ s.t. $\psi = \phi - \mathcal{C}_T\phi$.
But then $\exists \phi_0 \in D(T)$ s.t. $\phi = T\phi_0 + i\phi_0$
$\Rightarrow \mathcal{C}_T\phi = \mathcal{C}_T(T\phi_0 + i\phi_0) = T\phi_0 - i\phi_0$
$\Rightarrow \psi = T\phi_0 + i\phi_0 - T\phi_0 + i\phi_0 = 2i\phi_0 \in D(T)$.

Conversely, if $\psi_0 \in D(T)$, then $\phi := T\psi_0 + i\psi_0 \in D(C_T)$ and

$$C_T \phi = C_T(T\psi_0 + i\psi_0) = T\psi_0 - i\psi_0 .$$

$\Rightarrow$

$$\phi - C_T\phi = T\psi_0 + i\psi_0 - T\psi_0 + i\psi_0 = 2i\psi_0$$

$\Rightarrow$

$$\psi_0 = \frac{1}{2i}(\phi - C_T\phi) = (1 - C_T)\left(\frac{1}{2i}\phi\right) \in R(1 - C_T).$$

This proves $R(1-C_T) = D(T)$.

Also, if $(1-C_T)\phi = 0$ for some $\phi \in D(C_T)$,

$\Rightarrow$ $0 = \psi = 2i\phi_0$ in the above.

$\Rightarrow$ $\phi_0 = 0$ $\Rightarrow$ $\phi = T\phi_0 + i\phi_0 = 0$.

Thus $1-C_T$ is 1-1, and we have proven "a)".

To prove the "converse", let us assume V is an operator and an isometry, for which $1-V$ is 1-1. Then

$$(1-V)^{-1} : R(1-V) \to D(1-V) = D(-V) = D(V)$$
$$= D(1+V).$$

Thus we can define

$$T := i(1+V)(1-V)^{-1} : R(1-V) \to R(1+V).$$

Since $R(1\pm V)$ are subspaces and T is linear, (see the proof of Prop. 9.2.), T is an operator. Suppose $\psi, \phi \in D(T)$.

$\Rightarrow \exists\, \psi', \phi' \in D(V)$ s.t. $\psi = \psi' - V\psi'$
$\phi = \phi' - V\phi'$

In addition, $T\psi = i(\psi' + V\psi')$,
$T\phi = i(\phi' + V\phi')$.

$$\Rightarrow (\phi, T\psi) = (\phi' - V\phi', i(\psi' + V\psi'))$$
$$= i[(\phi', \psi') + (\phi', V\psi') - (V\phi', \psi') - (V\phi', V\psi')]$$

and
$$(T\phi, \psi) = -i(\phi' + V\phi', \psi' - V\psi')$$
$$= -i[(\phi', \psi') - (\phi', V\psi') + (V\phi', \psi') - (V\phi', V\psi')]$$

Therefore,

$$(\phi, T\psi) - (T\phi, \psi)$$
$$= 2i[(\phi', \psi') - (V\phi', V\psi')].$$

However, by the polarization identity (Exercise 3.7.),

$$(\phi', \psi') = \frac{1}{4}\Big(\|\phi'+\psi'\|^2 - \|\phi'-\psi'\|^2 - i\|\phi'+i\psi'\|^2 + i\|\phi'-i\psi'\|^2 \Big).$$

Since V is an isometric operator, we have $\forall \alpha \in \mathbb{C}$: $\phi' + \alpha\psi' \in D(V)$ and $\|\phi'+\alpha\psi'\| = \|V(\phi'+\alpha\psi')\|$ $= \|V\phi' + \alpha V\psi'\|$. Then a second application of the polar. identity shows that $(\phi', \psi') = (V\phi', V\psi')$.
Therefore, $(\phi, T\psi) = (T\phi, \psi)$ $\forall \phi, \psi \in D(T)$, and T is a symmetric operator.
For any $\psi \in D(T)$, $\exists \phi \in D(V)$ s.t.
$\psi = \phi - V\phi$ and $T\psi = i(\phi + V\phi)$.

$$\Rightarrow \quad T\psi + i\psi = i(\phi + V\phi + \phi - V\phi) = 2i\phi \in D(V)$$
$$T\psi - i\psi = i(\phi + V\phi - \phi + V\phi) = 2iV\phi \in R(V)$$

Thus $T\psi - i\psi = V(2i\phi) = V(T\psi + i\psi)$ $\forall \psi \in D(T)$, and $R(T+i) \subset D(V)$. Also $\phi \in D(V) \Rightarrow \psi := (1-V)\phi \in D(T)$ and $\phi = (T+i)(\frac{1}{2i}\psi) \Rightarrow \phi \in R(T+i)$. Thus $R(T+i) = D(V)$ and we can conclude that $V = \mathcal{C}_T$. This proves the "converse".
Let then T', T be symmetric operators. By o), we can apply the converse to the isometries $\mathcal{C}_T$ and $\mathcal{C}_{T'}$. If $\mathcal{C}_{T'} = \mathcal{C}_T$, by o), $D(T') = R(1-\mathcal{C}_{T'}) = R(1-\mathcal{C}_T) = D(T)$, and

$$T' = i(1+\mathcal{C}_{T'})(1-\mathcal{C}_{T'})^{-1} = i(1+\mathcal{C}_T)(1-\mathcal{C}_T)^{-1} = T.$$

This proves a).
For b), assume first $T \subset T'$. Then
$\psi \in D(\mathcal{C}_T) \Rightarrow \exists \phi \in D(T) \subset D(T')$ s.t.

$$\psi = T\phi + i\phi = T'\phi + i\phi \Rightarrow \psi \in D(\mathcal{C}_{T'}).$$

Also $\mathcal{C}_{T'}\psi = \mathcal{C}_{T'}(T'\phi + i\phi) = T'\phi - i\phi$
$= T\phi - i\phi = \mathcal{C}_T(T\phi + i\phi) = \mathcal{C}_T\psi$.
therefore, $\mathcal{C}_T \subset \mathcal{C}_{T'}$. For the converse, assume $\mathcal{C}_T \subset \mathcal{C}_{T'}$. Then $\psi \in D(T) = R(1-\mathcal{C}_T)$
$\Rightarrow \exists \phi \in D(\mathcal{C}_T) \subset D(\mathcal{C}_{T'})$ s.t. $\psi = \phi - \mathcal{C}_T\phi$
$= \phi - \mathcal{C}_{T'}\phi \Rightarrow \psi \in R(1-\mathcal{C}_{T'}) = D(T')$.

In addition, by the "converse", then

$$
\begin{aligned}
T'\psi &= i\,(1+\mathcal{C}_{T'})(1-\mathcal{C}_{T'})^{-1}\psi \\
&= i\,(1+\mathcal{C}_{T'})\phi = i\,(1+\mathcal{C}_T)\phi \\
&= i\,(1+\mathcal{C}_T)(1-\mathcal{C}_T)^{-1}\psi = T\psi .
\end{aligned}
$$

Therefore, then $T \subset T'$. This concludes the proof of b).

For c) and d), let us first note that, since $\mathcal{C}_T$ is an isometric operator, by Exercise 7.2.a) it has unique continuous extension $V : \overline{D(\mathcal{C}_T)} \to \overline{R(\mathcal{C}_T)}$ which is also an isometry. Clearly, $G(V) = \overline{G(\mathcal{C}_T)}$, and thus $\mathcal{C}_T$ is always closable, and $V = \overline{\mathcal{C}_T}$.

Assume first that $\mathcal{C}_T$ is closed $\Rightarrow V = \mathcal{C}_T$. As in (cc) on p. 43, let $\psi_n \in D(T)$ be a sequence for which $\psi_n \to \psi$ and $T\psi_n \to \phi$. Then $\phi_n := T\psi_n + i\psi_n \in D(\mathcal{C}_T)$ and $\phi_n \to \phi + i\psi$, and thus $\phi + i\psi \in \overline{D(\mathcal{C}_T)} = D(\mathcal{C}_T)$ and $\mathcal{C}_T\phi_n \to \mathcal{C}_T(\phi + i\psi)$.
But $\mathcal{C}_T\phi_n = \mathcal{C}_T(T\psi_n + i\psi_n) = T\psi_n - i\psi_n \to \phi - i\psi$. Therefore,

$$
\begin{aligned}
&\mathcal{C}_T(\phi + i\psi) = \phi - i\psi \\
&\Rightarrow 2i\psi = (1-\mathcal{C}_T)(\phi + i\psi) \in R(1-\mathcal{C}_T) \overset{a)}{=} D(T) \\
&\Rightarrow \psi \in D(T) \text{ and} \\
&\quad T\psi = i\,(1+\mathcal{C}_T)(1-\mathcal{C}_T)^{-1}\psi \\
&\qquad = \frac{i}{2i}(1+\mathcal{C}_T)(\phi + i\psi) = \frac{1}{2}(\phi + i\psi + \phi - i\psi) \\
&\qquad = \phi .
\end{aligned}
$$

Thus T satisfies (cc), and is closed.
For the converse, suppose T is closed. Consider $\phi \in \overline{D(\mathcal{C}_T)}$. Then there are $\phi_n \in D(\mathcal{C}_T)$ s.t. $\phi_n \to \phi$. $\Rightarrow \exists \psi_n \in D(T)$ s.t. $\phi_n = T\psi_n + i\psi_n$ and $\mathcal{C}_T\phi_n = T\psi_n - i\psi_n$. Thus

$$\psi_n = \frac{1}{2i}(\phi_n - \mathcal{C}_T\phi_n) \to \frac{1}{2i}(\phi - V\phi)$$

and $T\psi_n = \frac{1}{2}(\phi_n + \mathcal{C}_T\phi_n) \to \frac{1}{2}(\phi + V\phi)$.

Since T is closed, (cc) implies that

$$\frac{1}{2i}(\phi - V\phi) \in D(T) \;\&\; \frac{1}{2i}T(\phi - V\phi) = \frac{1}{2}(\phi + V\phi).$$

$\Rightarrow \phi - V\phi \in D(T)$ & $T(\phi - V\phi) = i(\phi + V\phi)$.
$\Rightarrow (T+i)(\phi - V\phi) = i(\phi + V\phi + \phi - V\phi) = 2i\phi$
$\Rightarrow \phi \in R(T+i) = D(C_T)$.
therefore, then $\overline{D(C_T)} = D(C_T) \Rightarrow V = C_T$ and C_T is closed. This proves c).

For d), assume T is symmetric and densely def. $\Rightarrow$ T is closable, and $\overline{T}$ is symmetric (Thm. 5.10. c)).
Thus by the above results $C_{\overline{T}}$ is closed and by b): $T \subset \overline{T} \Rightarrow C_T \subset C_{\overline{T}}$. $\Rightarrow \overline{C_T} \subset C_{\overline{T}}$.
On the other hand, $V = \overline{C_T}$ is an isometry, and if $\psi \in D(V)$ s.t. $(1-V)\psi = 0$
$\Rightarrow \psi \in D(C_{\overline{T}})$ and $(1 - C_{\overline{T}})\psi = 0 \Rightarrow \psi = 0$.
Thus $\exists \tilde{T} := i(1+V)(1-V)^{-1}$ which is a closed symmetric operator. Since $C_T \subset \overline{C_T} = C_{\tilde{T}} \subset C_{\overline{T}}$
$\overset{b)}{\Rightarrow} T \subset \tilde{T} \subset \overline{T}$. Therefore, $\tilde{T} = \overline{T}$
$\Rightarrow \overline{C_T} = C_{\tilde{T}} = C_{\overline{T}}$. This proves d). $\square$

9.5. Definition

Let S be densely defined and symmetric. Its *deficiency spaces* are $\mathcal{K}_+$ and $\mathcal{K}_-$, defined by

$$\mathcal{K}_+ := R(S+i)^\perp, \quad \mathcal{K}_- := R(S-i)^\perp.$$

The *deficiency indices* of S are

$$n_+ := \dim \mathcal{K}_+, \quad n_- := \dim \mathcal{K}_-$$

(Reminder: For a Hilbert space $\mathcal{H}$, $\dim \mathcal{H} = \mathrm{card}(ONB)$.)

9.6. Theorem Let S be a densely defined symmetric operator, and let $n_\pm$ denote its deficiency indices, and $\bar{S}$ its closure. Then one and only one of the following alternatives holds:

a) If $n_+ = n_- = 0$, S is essentially self-adjoint. (i.e. $\bar{S}$ is self-adjoint)

b) If $n_+ = 0$, $n_- \neq 0$ or $n_- = 0$, $n_+ \neq 0$, $\bar{S}$ is maximally symmetric, but not self-adjoint. Thus S has no self-adjoint extensions.

c) If $n_+ \neq n_-$, and $n_+, n_- \neq 0$, then S has symmetric extensions, but no self-adjoint ones.

d) If $n_+ = n_- \neq 0$, S is not essentially self-adjoint, but it has infinitely many self-adjoint extensions.
If $W: \mathcal{K}_+ \to \mathcal{K}_-$ is Hilbert space isomorphism, then there is a unique self-adj. extension A of S such that

$$\mathcal{C}_A = \overline{\mathcal{C}_S} \oplus W \quad \text{(Notation: see Ex. 7.2)}$$

In addition, every self-adjoint extension of S can be obtained this way.

9.7. Corollary: A densely def. symmetric operator S has self-adjoint extensions if and only if $n_+ = n_-$.

If A is a self-adjoint extension, with Cayley transf. $\mathcal{C}_A = \overline{\mathcal{C}_S} \oplus W$, then $D(A) = \{ \psi + \varphi_+ - W\varphi_+ \mid \psi \in D(\bar{S}),\ \varphi_+ \in \mathcal{K}_+ \}$ and $\forall\, \psi \in D(\bar{S}),\ \varphi_+ \in \mathcal{K}_+$ we have

$$A(\psi + \varphi_+ - W\varphi_+) = \bar{S}\psi + i\varphi_+ + iW\varphi_+ .$$

For theorem 9.6., we need the following Lemma:

9.8. Lemma Suppose T is a densely defined closed operator on $\mathcal{H}$. Then for any $\phi, \phi' \in \mathcal{H}$ there are unique $\psi \in D(T)$, $\psi' \in D(T^*)$ s.t.

$$(*) \begin{cases} -T\psi + \psi' = \phi \\ \psi + T^*\psi' = \phi' \end{cases}$$

Proof: Consider the proof of Theorem 5.7. There we defined the map $V((\psi, \phi)) = ((-\phi, \psi))$, which was unitary on $\mathcal{H} \oplus \mathcal{H}$, and showed that $\mathcal{G}(T^*) = V(\mathcal{G}(T))^\perp$. Also since now $\mathcal{G}(T)$ is closed, also $V(\mathcal{G}(T))$ is a closed subspace, and thus by Ex. 3.2. $\Rightarrow V\mathcal{G}(T) = [V(\mathcal{G}(T))^\perp]^\perp = \mathcal{G}(T^*)^\perp$. Since $\mathcal{G}(T^*)$ is also a closed subspace, this implies (Theorem 2.11.) that

$$\mathcal{H} \oplus \mathcal{H} = \mathcal{G}(T^*) \oplus \mathcal{G}(T^*)^\perp = \mathcal{G}(T^*) \oplus V(\mathcal{G}(T)).$$

Thus for any $((\phi, \phi')) \in \mathcal{H} \oplus \mathcal{H}$ there are unique $a \in \mathcal{G}(T^*)$, $b \in V(\mathcal{G}(T))$ s.t. $((\phi, \phi')) = a + b$.
$\Rightarrow \exists \psi \in D(T)$ and $\psi' \in D(T^*)$ s.t.
$a = ((\psi', T^*\psi'))$, $b = ((-T\psi, \psi))$.
$\Rightarrow (*)$ holds. To see uniqueness, assume also $\tilde{\psi}$, and $\tilde{\psi}'$ satisfy $(*)$, and def.
$\tilde{a} = ((\tilde{\psi}', T^*\tilde{\psi}'))$, $\tilde{b} = ((-T\tilde{\psi}, \tilde{\psi})) = V((\tilde{\psi}, T\tilde{\psi}))$
Then $\tilde{a} \in \mathcal{G}(T^*)$, $\tilde{b} \in V(\mathcal{G}(T))$ and
$\tilde{a} + \tilde{b} = ((\phi, \phi'))$. $\Rightarrow \tilde{a} = a$, $\tilde{b} = b$ □

9.9. Corollary Suppose T is densely defined and closed operator. Then
$$R(1 + T^*T) = \mathcal{H}.$$

Proof: Suppose $\phi' \in \mathcal{H}$ is given, and apply the lemma with $\phi = 0$. $\Rightarrow \exists \psi \in D(T)$, $\psi' \in D(T^*)$ s.t. $\psi' = T\psi$ and
$\phi' = \psi + T^*\psi' = \psi + T^*T\psi$
$\Rightarrow \psi \in D(1 + T^*T)$ and $\phi' = (1 + T^*T)\psi$ □

9.10. Lemma: Suppose A is a symmetric op. Then A is self-adjoint $\Leftrightarrow$ e_A is unitary.

Proof: Suppose first A is self-adjoint. By Corollary 9.9., $D(1+A^*A) = D(A^2)$
and $R(1+A^2) = \mathcal{H}$.

For any $\psi \in D(A^2)$ we have
$$(1+A^2)\psi = (A+i)(A-i)\psi = (A-i)(A+i)\psi$$
and thus $R(A+i) = \mathcal{H} = R(A-i)$.

This implies that e_A is an isometry with $\underbrace{R(e_A) = \mathcal{H}}_{= D(e_A)}$

$\Rightarrow$ e_A is unitary (Ex. 3.4.)

Conversely, assume A is symmetric and e_A is unitary. Then if $\phi \perp R(1-e_A)$
$$\Rightarrow \forall \psi \in \mathcal{H}: (\phi, (1-e_A)\psi) = 0$$
$$= (\phi, \psi) - (\phi, e_A\psi) = (\phi, \psi) - (e_A^*\phi, \psi)$$
$$= (\phi - e_A^*\phi, \psi)$$
$\Rightarrow$ $\phi = e_A^*\phi \Rightarrow e_A\phi = \phi$. But since $1-e_A$ is injective (Thm. 9.4.0))

$\Rightarrow$ $\phi = 0$. Thus $R(1-e_A)^\perp = D(A)^\perp = \{0\}$

$\Rightarrow$ $\overline{D(A)} = (D(A)^\perp)^\perp = \mathcal{H}$. Therefore,

A is then densely defined. $\Rightarrow \exists A^*$.

Since A is symmetric, then $A \subset A^*$.
(Ex. 5.4.) Let $\phi \in D(A^*) (\supset D(A))$.

Since $R(A+i) = D(e_A) = \mathcal{H}$

$\Rightarrow \exists \tilde{\phi} \in D(A)$ s.t. $(A^*+i)\phi = (A+i)\tilde{\phi}$
$\underset{A \subset A^*}{=} (A^*+i)\tilde{\phi}$. Let $\psi_0 := \phi - \tilde{\phi} \in D(A^*)$.

Then $\forall \psi \in D(A)$:
$$(\psi_0, (A-i)\psi) = (\overbrace{(A^*+i)\psi_0}^{=0}, \psi) = 0$$
$$\Rightarrow \psi_0 \in R(A-i)^\perp = R(e_A)^\perp = \mathcal{H}^\perp = \{0\}$$
$$\Rightarrow \psi_0 = 0 \text{ and } \phi = \tilde{\phi} \Rightarrow \phi \in D(A)$$
$$\Rightarrow A^*\phi = A\phi.$$
Therefore, also $A^* \subset A \Rightarrow A^* = A$ and A is self-adjoint. $\square$

Proof of Theorem 9.6.

a) If $n_+ = n_- = 0 \Rightarrow \mathcal{K}_+ = \{0\} = \mathcal{K}_-$
$\Rightarrow \overline{R(\bar{S}+i)} = \mathcal{H} = \overline{R(\bar{S}-i)}$.
By Thm. 9.4. $\mathcal{C}_{\bar{S}} = \overline{\mathcal{C}_S}$
$\Rightarrow D(\mathcal{C}_{\bar{S}}) = \overline{R(S+i)} = \mathcal{H}$
$R(\mathcal{C}_{\bar{S}}) = \overline{R(S-i)} = \mathcal{H}$.
Thus $\mathcal{C}_{\bar{S}}$ is unitary (Ex. 3.4.)
$\Rightarrow \bar{S}$ is self-adjoint. (Lemma 9.10.)

Suppose then, that $A \supset S$, and A is self-adjoint. By Thm. 9.4. & Lemma 9.10.
$\Rightarrow \mathcal{C}_S \subset \mathcal{C}_A =$ unitary. Thus $\mathcal{C}_A$ is then a unitary extension of the isometry $\mathcal{C}_S$.
Exercise 7.2. b) $\Rightarrow$ $W = (\mathcal{C}_A - (\overline{\mathcal{C}_S} \oplus 0))|_{D(\mathcal{C}_S)^\perp}$ is an isomorphism $\underbrace{D(\mathcal{C}_S)^\perp}_{= R(S+i)^\perp} \to \underbrace{R(\mathcal{C}_S)^\perp}_{= R(S-i)^\perp}$
i.e. between $\mathcal{K}_+$ and $\mathcal{K}_-$.
If $n_+ \neq n_-$, $\mathcal{K}_+$ and $\mathcal{K}_-$ are not isomorphic, and thus S then cannot have any self-adjoint extensions.
$\Rightarrow \bar{S}$ is not self-adjoint.
If $n_+ = n_-$ and $W: \mathcal{K}_+ \to \mathcal{K}_-$ is an isomorphism, then $U := \overline{\mathcal{C}_S} \oplus W = \mathcal{C}_{\bar{S}} \oplus W$ is unitary on $\mathcal{H}$. Suppose $\psi \in \mathcal{H}$ is such that $(1-U)\psi = 0$. As $\mathcal{H} = D(\mathcal{C}_{\bar{S}}) \oplus \mathcal{K}_+$
$\Rightarrow \exists!\ \psi_0 \in D(\mathcal{C}_{\bar{S}})$, $\psi_+ \in \mathcal{K}_+$ s.t.
$\psi = \psi_0 + \psi_+$. In addition, by def.,
$U\psi = \mathcal{C}_{\bar{S}}\psi_0 + W\psi_+$. Let $\psi_- = W\psi_+ \in \mathcal{K}_-$,
when $0 = \psi - U\psi = (1-\mathcal{C}_{\bar{S}})\psi_0 + \psi_+ - \psi_-$
$\Rightarrow \psi_- = \psi_+ + \phi_0$, where $\phi_0 := (1-\mathcal{C}_{\bar{S}})\psi_0$
$\in R(1-\mathcal{C}_{\bar{S}}) = D(\bar{S})$. Since $\psi_- \in \mathcal{K}_-$
$= R(S-i)^\perp = \overline{R(S-i)}^\perp = R(\mathcal{C}_{\bar{S}})^\perp = R(\bar{S}-i)^\perp$

$$\Rightarrow \forall \phi \in D(\bar{S}):\ 0 = ((\bar{S}-i)\phi, \psi_-)$$
$$= ((\bar{S}-i)\phi, \phi_0 + \psi_+)$$
$\bar{S}$ symm.
$$= (\phi, (\bar{S}+i)\phi_0) + (\underbrace{(\bar{S}+i)\phi}_{\in R(\bar{S}+i)} - 2i\phi, \underbrace{\psi_+}_{\in \mathcal{K}_+ = \overline{R(S+i)}^\perp = R(\bar{S}+i)^\perp})$$
$$= (\phi, (\bar{S}+i)\phi_0) + (\phi, 2i\psi_+)$$
$$= (\phi, (\bar{S}+i)\phi_0 + 2i\psi_+)$$

$D(\bar{S})$ is dense!

$\overset{!}{\Rightarrow} \quad (\bar{S}+i)\phi_0 + 2i\psi_+ = 0$

$\Rightarrow \psi_+ = \frac{1}{-2i}(\bar{S}+i)\phi_0 \in R(\bar{S}+i)$

$\Rightarrow \psi_+ \in R(\bar{S}+i) \cap \mathcal{K}_+ = R(\bar{S}+i) \cap R(\bar{S}+i)^\perp$

$\Rightarrow \psi_+ = 0 \Rightarrow \psi_- = W\psi_+ = 0$

$\Rightarrow 0 = \psi - U\psi = (1 - C_{\bar{S}})\psi_0$.

Since $\bar{S}$ is symm. op. $\Rightarrow 1 - C_{\bar{S}}$ is 1-1.

$\Rightarrow$ also $\psi_0 = 0$. Thus $\psi = 0$.

This proves that $1-U$ is 1-1.

Thm. 9.4.

$\overset{!}{\Rightarrow} \quad A = i(1+U)(1-U)^{-1}$ is symm. oper. and $U = C_A$.

Since U is unitary, Lemma 9.16.

$\Rightarrow A$ is self-adjoint.

We have thus proven d). (Note that $\exists$ inf. many maps W, see p. 107)

To prove b) & c), assume $n_+ \neq n_-$, and suppose S' is a symm. extension of $\bar{S}$. $\Rightarrow C_{\bar{S}} \subset C_{S'}$. If $n_+ = 0$, we have $\mathcal{K}_+ = \{0\} \Rightarrow R(\bar{S}+i) = \mathcal{H} = D(C_{\bar{S}})$

$\Rightarrow C_{S'} = C_{\bar{S}} \Rightarrow S' = \bar{S}$. If $n_- = 0 \Rightarrow \mathcal{K}_- = \{0\}$

$\Rightarrow R(C_{\bar{S}}) = \mathcal{H}$. Suppose $\psi \in D(C_{S'})$ is such that $\psi \in \mathcal{K}_+ = D(C_{\bar{S}})^\perp \Rightarrow$

$C_{S'}\psi \in R(C_{\bar{S}}) \Rightarrow \exists \psi' \in D(C_{\bar{S}})$ s.t.

$C_{S'}\psi = C_{\bar{S}}\psi' = C_{S'}\psi' \Rightarrow C_{S'}(\psi - \psi') = 0$

$\xRightarrow{C_{S'} \text{ is } 1-1} \psi - \psi' = 0 \Rightarrow \psi' = \psi$

$\Rightarrow \psi \in D(C_{\bar{S}})^\perp \cap D(C_{\bar{S}}) \Rightarrow \psi = 0$.

Thus $D(C_{S'}) = \overline{D(C_{\bar{S}})} = D(C_{\bar{S}}) \Rightarrow C_{S'} = C_{\bar{S}} \Rightarrow S' = \bar{S}$.

This proves that $\bar{S}$ is maximally symmetric, and concludes the proof b).

If $n_+ \neq n_-$, $n_+, n_- \neq 0 \Rightarrow \exists e_+ \in \mathcal{K}_+, e_- \in \mathcal{K}_-$ s.t. $\|e_\pm\| = 1$. Let $R_+ := R(\bar{S}+i) \oplus \mathrm{span}(e_+)$,

$R_- := R(\bar{S}-i) \oplus \mathrm{span}(e_-)$ and define

$V : R_+ \to R_-$ by $V(\psi + \alpha e_+) = C_{\bar{S}}\psi + \alpha e_-$

$\forall \alpha \in \mathbb{C}, \psi \in D(C_{\bar{S}}) = R(\bar{S}+i)$. Then V is an isometry, and a similar argument to the above case proves that $1 - V$ is injective. $\Rightarrow \exists S'$ s.t. $V = C_{S'}$. Since $C_S \subsetneq V \Rightarrow S \subsetneq S'$. $\square$

Proof of Corollary 9.7: Suppose S is dens. def. and symmetric.
By Thrm 9.6. it has self-adj. extensions iff $n_+ = n_-$. Moreover, if A is self-adj. and $S \subset A$, then $\mathcal{C}_A = \overline{\mathcal{C}_S} \oplus W$. (set $W=0$, if $n_+ = n_- = 0$), where $W: \mathcal{K}_+ \to \mathcal{K}_-$ is a unitary map.

For simplicity, denote $U := \mathcal{C}_A \in \mathcal{B}(\mathcal{H})$ and $V := \overline{\mathcal{C}_S}$. By Theorem 9.4., then $V = \mathcal{C}_{\overline{S}}$ and

$$A = i(1+U)(1-U)^{-1} \quad \text{with } D(A) = R(1-U).$$

Since $D(U) = \mathcal{H} = \overline{D(\mathcal{C}_S)} \oplus \mathcal{K}_+$, if $\psi \in D(A)$ then $\exists\, \phi_0 \in \overline{D(\mathcal{C}_S)} = D(V)$ and $\varphi_+ \in \mathcal{K}_+$ s.t. $\phi_0 \perp \varphi_+$ and $\psi = (1-U)(\phi_0 + \varphi_+) = \phi_0 - V\phi_0 + \varphi_+ - W\varphi_+$
$= (1-V)\phi_0 + \varphi_+ - W\varphi_+$. Here $(1-V)\phi_0 \in R(1-V) = D(\overline{S})$
$\Rightarrow \psi \in D_0 := \{ \psi_0 + \varphi_+ - W\varphi_+ \mid \psi_0 \in D(\overline{S}),\ \varphi_+ \in \mathcal{K}_+ \}$.
If $\psi \in D_0 \Rightarrow \exists\, \psi_0 \in D(\overline{S}) = R(1-V)$, $\varphi_+ \in \mathcal{K}_+$ s.t.
$\psi = \psi_0 + \varphi_+ - W\varphi_+ \Rightarrow \exists\, \phi_0 \in D(V)$ s.t. $\psi = (1-V)\phi_0 + \varphi_+ - W\varphi_+$
$= \phi_0 + \varphi_+ - V\phi_0 - W\varphi_+ = (1-U)(\phi_0 + \varphi_+) \in R(1-U) = D(A)$.
Then also $A\psi = i(1+U)(\phi_0 + \varphi_+) = i(1+U)\phi_0 + i\varphi_+ + iU\varphi_+$
$= i(1+V)\phi_0 + i\varphi_+ + iW\varphi_+$, where $i(1+V)\phi_0 = i(1+V)(1-V)^{-1}(1-V)\phi_0$
$= \overline{S}(1-V)\phi_0 = \overline{S}(1-U)\phi_0 = \overline{S}\psi_0$

Therefore, $D(A) = D_0$ and $\psi \in D(A) \Rightarrow \exists\, \psi_0 \in D(\overline{S}), \varphi_+ \in \mathcal{K}_+$ s.t. $\psi = \psi_0 + \varphi_+ - W\varphi_+$. Whatever the choice of ψ_0, φ_+, then also

$$A\psi = A(\psi_0 + \varphi_+ - W\varphi_+) = \overline{S}\psi_0 + i\varphi_+ + iW\varphi_+ \quad \square$$

* For instance, if $n_+ = n_- = n < \infty$, then $W: \mathcal{K}_+ \to \mathcal{K}_-$ are in one-to-one correspondence with matrices $w \in U(n)$ (= set of unitary $n \times n$ matrices.) Explicitly, if $\{e_-^i\}$ is an ONB for $\mathcal{K}_-$ and $\{e_+^j\}$ for $\mathcal{K}_+$, then (proof left as a straightforward exercise)

$$W: \mathcal{K}_+ \to \mathcal{K}_- \text{ is a unitary map} \iff \exists!\, w \in U(n) \text{ s.t. } W\varphi = \sum_{i,j=1}^{n} e_-^i\, w_{ij}\, (e_+^j, \varphi) \ \forall \varphi \in \mathcal{K}_+.$$

Hence, any parametrization of $U(n)$ yields a parametrization of the self-adjoint extensions. $\Rightarrow$ inf. many, if $n > 0$.