8.4. Application II: Weyl-quantization and symbolic calculus with pseudo-differential operators

1. Proposition: The mapping $f \mapsto W[f]$

defined for $f \in S(\mathbb{R}^d \times \mathbb{R}^d)$ by

$$W[f](x,k) := \int_{\mathbb{R}^d} dy\; e^{-i2\pi k\cdot y} f\left(x-\frac{y}{2}, x+\frac{y}{2}\right)$$

is a continuous, linear map $S_{2d} \to S_{2d}$.

Proof. Extra exercise. For instance, prove that $\|W[f]\|_{S,N} \leq C^N \|f\|_{S,N+2d}$ $\forall N$. $\square$

2. Definition The Wigner transform of a distribution $\Lambda \in S'(\mathbb{R}^d \times \mathbb{R}^d)$

is the tempered distribution $\Lambda^W \in S'(\mathbb{R}^d \times \mathbb{R}^d)$ defined by $\Lambda^W(f) := \Lambda(W[f])$.

3. Definition For $\Lambda \in S'(\mathbb{R}^d \times \mathbb{R}^d)$ define

$Q_\Lambda : S \times S \to \mathbb{C}$ by

$$Q_\Lambda(\phi, \psi) := \Lambda^W(\phi^* \otimes \psi)$$

where $(\phi^* \otimes \psi)(x_1, x_2) := \phi(x_1)^* \psi(x_2)$.
$\in S(\mathbb{R}^d \times \mathbb{R}^d)$.

If there is a closed operator T on $L^2(\mathbb{R}^d)$, such that

a) S_d is a core for T
$\Leftrightarrow$ $S_d \subset D(T)$ and $\overline{T|_S} = T$.

b) $\forall \phi, \psi \in S_d$: $Q_\Lambda(\phi,\psi) = (\phi, T\psi)$,

Then Λ is Weyl-quantizable and T is a Weyl quantization of Λ.

4. Definition A particular case is the quantization of symbols:

$a(x,k)$; $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$,

with $|\partial_x^\alpha \partial_k^\beta a(x,k)| \le C_{\alpha,\beta}\,(1+|k|)^{m-|\beta|}$
for some $m \in \mathbb{Z}$, and all α, β, x, k.

The Weyl-quantization of

$$\Lambda(f) = \int dx\,dk\; a(x,k)\, f(x,k)$$

is then called a (Weyl-quantized) pseudo-differential operator, and denoted by $a^W(x, \frac{1}{2\pi i}\partial)$.

This procedure can be used to give meaning to "quantization" of essentially all smooth classical Hamiltonians, such as with electro-magnetic fields.

For references on the topic, see for instance the book

L. Hörmander: The Analysis of Linear Partial Differential Operators III: Pseudo-Differential operators, Springer, 1994.

5. Proposition Let a be a symbol and $\hat{a}^W$ its Weyl-quantization (if it exists).

a) $a(x,k) = x_\nu \Rightarrow \hat{a}^W = M_{x_\nu}$
$= \text{multip. by } x_\nu$

b) $a(x,k) = 2\pi k_\nu \Rightarrow \hat{a}^W = \hat{p}_\nu$
$= -i\partial_\nu$.

Proof: Exercise □

6. Other quantization schemes

In Weyl quantization, functions $a(x,k)$ on the phase space are connected with their quantizations by the rule

$$(*) \quad \langle \phi | \hat{a}^W \psi \rangle = \int dx\, dk\; a(x,k)\, W[\phi,\psi](x,k).$$

where $W[\phi,\psi](x,k) := \int dy\, e^{-i2\pi k\cdot y} \phi(x-\frac{1}{2}y)^* \psi(x+\frac{y}{2})$.

Although convenient for many situations involving quantum systems, it is not the only possibility how to map phase space observables to operators.

Other common definitions include a real parameter $\tau \in \mathbb{R}$ using which one defines τ-Wigner functions by setting

$$W_\tau[\phi,\psi](x,k) := \int_{\mathbb{R}^d} dy\, e^{-i2\pi k\cdot y} \phi(x-(1-\tau)y)^* \psi(x+\tau y)$$

and then defining $\hat{a}_\tau$ by replacing W with W_τ in (*).

* $\tau = 1/2$ corresponds to Weyl quantization
* $\tau = 0$ corresponds to Kohn-Nirenberg quantization which is often used to define "pseudo-differential operators" in mathematics. (E.g. in Hörmander's books)

[More details about τ-quantization can be found in Chapter 4 of the book:
M. A. Shubin, Pseudodifferential Operators and Spectral Theory, Springer, 1980.

See also Hall's book, chapter 13.]

One more generalization which leads to somewhat more complicated analysis but also improves regularity properties is called <u>Born-Jordan quantization</u>: Instead of $W[\phi, \psi]$, to get the quantized $\hat{a}^{Q}$, one uses in $(*)$ the average of W_τ-functions defined by

$$Q[\phi, \psi](x,k) := \int_0^1 d\tau \; W_\tau[\phi, \psi](x,k) .$$

* This version removes some "ghost-frequencies" which often appear when using the functions W_τ such as the Wigner function.

* Particular useful in one-dimensional analysis ($d=1$ above) $\Rightarrow$ used now in numerical <u>time-frequency analysis</u> [Contact Ville Turunen at Aalto University for more details.]

[More details about the Born-Jordan quantization: M. de Gosson, F. Luef, Preferred quantization rules: Born-Jordan versus Weyl. The pseudo-differential point of view, J. Pseudo-Differ. Oper. Appl. (2011) 115. <u>or</u> arXiv.org: 1102.5732]

* Examples of different quantizations above ($d=1$)

1) If $a(x,p) = xp$, then

$$\hat{a}^{W} = \frac{1}{2}(\hat{x}\hat{p} + \hat{p}\hat{x})$$
$$\hat{a}_\tau = \tau \hat{p}\hat{x} + (1-\tau)\hat{x}\hat{p} \quad \Rightarrow \quad \hat{a}_0 = \hat{x}\hat{p}$$

$$\hat{a}^{Q} = \frac{1}{2}(\hat{x}\hat{p} + \hat{p}\hat{x}) = \hat{a}^{W}$$

2) If $a(x,p) = xp^2$, then $\hat{a}_0 = \hat{x}\hat{p}^2$,

$$\hat{a}^{W} = \frac{1}{4}(\hat{p}^2\hat{x} + 2\hat{p}\hat{x}\hat{p} + \hat{x}\hat{p}^2) \neq \hat{a}^{Q} = \frac{1}{3}(\hat{p}^2\hat{x} + \hat{p}\hat{x}\hat{p} + \hat{x}\hat{p}^2)$$