13. Fermionic lattice systems

As an easy, but nontrivial, example consider a one-particle space $h := l_2(\mathbb{Z}^d)$. The vectors $(e_x)_{x\in\mathbb{Z}^d}$ defined by $e_x(y) := \mathbb{1}(x=y)$, $y\in\mathbb{Z}^d$, obviously form an ONB for h. In this case, it is customary to denote

$$a(x) := a_-(e_x) \Rightarrow a(x)^* = (a_-(e_x))^* = a_-^*(e_x).$$

Since $\|a(x)\| = \|e_x\| = 1 = \|a(x)^*\|$ $\forall x\in\mathbb{Z}^d$, (Thm. 12.4.1.1.) then for any $v: \mathbb{Z}^d\times\mathbb{Z}^d \to \mathbb{C}$ such that $\sum_{x,y\in\mathbb{Z}^d} |v(x,y)| < \infty$ (i.e. $v\in l_1(\mathbb{Z}^d\times\mathbb{Z}^d)$) the following operators are well-defined as convergent sums in $\mathcal{B}(\mathcal{F}^{(-)})$:

$$B_1 := \sum_{x,y\in\mathbb{Z}^d} v(x,y)\, a(x)^* a(y)$$

$$B_2 := \sum_{x,y\in\mathbb{Z}^d} v(x,y)\, a(x)^* a(y)^* a(y)\, a(x)$$

In addition, since "$*$" is continuous, it follows that B_1 is self-adjoint, if $v(x,y)^* = v(y,x)$ $\forall x,y$ and B_2 is self-adjoint, if $v(x,y)\in\mathbb{R}$ $\forall x,y$.

The following lemma implies that B_2 corresponds to a multiplication operator which is given by a two-body potential:

13.1. Lemma: If $l\in I_0$, and $N := |l| < \infty$, then $\forall\, l_0\in I$:

$$a_-^*(e_{l_0})\, a_-(e_{l_0})\, a_-^*(e_{l_1})\cdots a_-^*(e_{l_N})\Omega$$

$$= \left(\sum_{n=1}^{N} \mathbb{1}(l_0 = l_n)\right) a_-^*(e_{l_1})\cdots a_-^*(e_{l_N})\Omega.$$

Proof: Induction in N. If $N=0$
$\Rightarrow a_-^*(e_{l_0})\, a_-(e_{l_0})\Omega = 0$, ok.

Assume then that the result hols for $l'; |l'|<N$. Then by the anticommutation relations (12.4.1.4.)

$$(*)\quad a_-^*(e_{l_0})a_-(e_{l_0})a_-^*(e_{l_1}) = a_-^*(e_{l_0})\left((e_{l_0},e_{l_1})\mathbb{1} - a_-^*(e_{l_1})a_-(e_{l_0})\right) = \mathbb{1}(l_0=l_1)\,a_-^*(e_{l_0}) + a_-^*(e_{l_1})a_-^*(e_{l_0})a_-(e_{l_0})$$

$\Rightarrow$

$$a_-^*(e_{l_0})a_-(e_{l_0})a_-^*(e_{l_1})\cdots a_-^*(e_{l_N})\Omega = \mathbb{1}(l_0=l_1)\,a_-^*(e_{l_1})a_-^*(e_{l_2})\cdots a_-^*(e_{l_N})\Omega + a_-^*(e_{l_1})\Big(\sum_{n=2}^{N}\mathbb{1}(l_0=l_n)\Big)a_-^*(e_{l_2})\cdots a_-^*(e_{l_N})\Omega = \sum_{n=1}^{N}\mathbb{1}(l_0=l_n)\,a_-^*(e_{l_1})\cdots a^*(e_{l_N})\Omega \quad \square$$

* An identical proof using the commutation relations (12.4.2.1.e) shows also that $\forall\, l \in I_0$:

$$\tilde{c}(e_{l_0})\tilde{a}(e_{l_0})e(l) = \sum_{n=1}^{|l|}\mathbb{1}(l_n=l_0)\cdot e(l).$$

(Instead of "(*)" above, we have then

$$\tilde{c}(e_{l_0})\tilde{a}(e_{l_0})\tilde{c}(e_{l_1}) = \tilde{c}(e_{l_0})\left(\mathbb{1}(l_0=l_1)\mathbb{1} + \tilde{c}(e_{l_1})\tilde{a}(e_{l_0})\right) = \mathbb{1}(l_0=l_1)\,\tilde{c}(e_{l_1}) + \tilde{c}(e_{l_1})\tilde{c}(e_{l_0})\tilde{a}(e_{l_0}).$$)

* Since $a(x)^*a(y)^*a(y)a(x) = 0$, if $x=y$, and if $x\neq y$, it is $= a(x)^*a(x)a(y)^*a(y)$, the Lemma implies that $\forall l \in I^{(-)}$:

$$B_2 e(l) = \sum_{\substack{x,y\in\mathbb{Z}^d \\ x\neq y}} v(x,y)\sum_{i,j=1}^{N}\mathbb{1}(x=l_i)\,\mathbb{1}(y=l_j)\,e(l)$$

$$= \sum_{i,j=1}^{N}\mathbb{1}(l_i\neq l_j)\,v(l_i,l_j)\cdot e(l) \Rightarrow \text{if } x\in I^{(-)} \text{ and } N=|x|,$$

$$\Rightarrow (B_2\Psi)_N(x) := \Big(\underbrace{\bigotimes_{n=1}^{N} e_{\bar{x}_n}}_{\in\mathcal{F}_N^{(-)}}, (B_2\Psi)_N\Big) = \Big(P_N^{(-)}\big(\otimes e_{\bar{x}_n}\big), (B_2\Psi)_N\Big)$$

$$= \frac{1}{\sqrt{N!}}\left(e(x), B_2\Psi\right) = \sum_{y',y\in\mathbb{Z}^d} v(y',y)\frac{1}{\sqrt{N!}}\left(a(y')^*a(y)^*a(y)a(y')e(x), \Psi\right)$$

$$= \underbrace{\Big(\sum_{n',n=1}^{N}\mathbb{1}(\bar{x}_{n'}\neq\bar{x}_n)\,v(\bar{x}_{n'},\bar{x}_n)\Big)}_{=\text{ 2-body potential; compare to p. 127.}}\cdot\Psi_N(x)$$

* If $v(x,y) = 0 \ \forall x \neq y$, similarly it follows that
$$(B_1 \Psi)_N(x) = \sum_{n=1}^{N} v(\bar{x}_n, \bar{x}_n) \cdot \Psi_N(x).$$

* In general, $a(y)e(\ell) = \sum_{n=1}^{|\ell|} (-1)^{n-1} \mathbb{1}(\ell_n = y)\, e(\hat{\ell}^{(n)})$
where $\hat{\ell}^{(n)}$ = "ℓ omit ℓ_n" as before.
$\Rightarrow a(x)^* a(y) e(\ell) = \sum_{n=1}^{|\ell|} (-1)^{n-1} \mathbb{1}(\ell_n = y)\, e((x, \hat{\ell}^{(n)}))$
$= \sum_{n=1}^{|\ell|} \mathbb{1}(\ell_n = y)\, e(\ell^{(n,x)})$ where $\ell^{(n,x)}$ = "ℓ replace ℓ_n by x".
Thus, if $x \in (\mathbb{Z}^d)^N$, we have
$$(B_1 \Psi)_N(x) = \sum_{y', y \in \mathbb{Z}^d} v(y', y) \frac{1}{\sqrt{N!}} (e(x), a(y')^* a(y) \Psi)$$
$$= \sum_{y', y} v(y', y) (a(y)^* a(y') e(x), \Psi) \frac{1}{\sqrt{N!}}$$
$$= \sum_{n=1}^{N} \sum_{y', y} v(y', y)\, \mathbb{1}(\bar{x}_n = y') \frac{1}{\sqrt{N!}} (e(x^{(n,y)}), \Psi)$$
$$= \sum_{n=1}^{N} \sum_{y \in \mathbb{Z}^d} v(\bar{x}_n, y)\, \Psi_N(x^{(n,y)})$$
$= \sum_{n=1}^{N} (v^{(n)} \Psi_N)(x)$. Thus B_1 is similar to H_0, with $v^{(n)}$ replacing $-\frac{1}{2}\nabla^2$.

* If $(U_t)_{t \geq 0}$ is a unitary semigroup on $\mathcal{F}^{(-)}$, one can equivalently study the evolution of
$a(t,x) := U_t^* a_-(x) U_t$ $\left(\Rightarrow a(t,x)^* = U_t^* a_-^*(x) U_t . \right)$
since then $\forall \Psi_0 \in \mathcal{F}^{(-)}$, $x \in (\mathbb{Z}^d)^N$:
$$\Psi_N(t,x) = \frac{1}{\sqrt{N!}} (e(x), U_t \Psi_0) = \frac{1}{\sqrt{N!}} (U_t^* e(x), \Psi_0)$$
where $U_t^* e(x) = U_t^* a(\bar{x}_1)^* \overset{U_t U_t^*}{\cdots} a(\bar{x}_N)^* \overset{U_t U_t^*}{} \Omega$
$= a(t, \bar{x}_1)^* \cdots a(t, \bar{x}_n)^* U_t^* \Omega$
and often $U_t \Omega = \Omega = U_t^* \Omega$.
If $U_t = e^{-itH}$, with H bounded, then
$\partial_t a(t,x) = e^{itH} (iH a(x) - i a(x) H) e^{-itH}$
$= i e^{itH} [H, a(x)] e^{-itH}$

Thus, for instance, for a 1-body potential B_1:
$$[B_1, a_-(x)] = \sum_{y', y} v(y', y) \underbrace{[a(y')^* a(y), a(x)]}_{= -\mathbb{1}(x = y') a(y)}$$
$= -\sum_y v(x,y) a(y) \Rightarrow \partial_t a(t,x) = -i \sum_y v(x,y)\, a(t,y).$

* For B_2, one gets a similar non-linear equation for $a(t, \cdot)$.