12.4.2. Bosonic creation and annihilations operators

It turns out that the bosonic operators are not bounded, but neither are they normal. Hence their definition requires some additional technicalities.

We begin by showing that the restrictions have many properties which are are commuting analogues of the fermionic ones:

1. Theorem: For any $g \in h$, define

$$D_+^0 := \{ \Psi \in \mathcal{F}^{(+)} \mid \exists N_0 \in \mathbb{N}_0 \text{ s.t. } \Psi_N = 0 \ \forall N \geq N_0 \}$$

and $\tilde{a}(g) := P^{(+)} a(g)|_{D_+^0}$ and $\tilde{c}(g) := P^{(+)} c(g)|_{D_+^0}$.

Then $\tilde{a}(g)$ and $\tilde{c}(g)$ are densely defined operators on $\mathcal{F}^{(+)}$. In addition,

o) $g \mapsto \tilde{a}(g)$ is conj. lin. and $g \mapsto \tilde{c}(g)$ is linear,

a) Any finite monomial of such $\tilde{a}, \tilde{c}$ is an operator on D_+^0.

b) For $N \in \mathbb{N}_+$, $g \in h^N$, set $\Psi(g) \in \mathcal{F}^{(0)}$ as in Prop. 12.4.1.2, $\Psi(g)_N = \bigotimes_{n=1}^{N} g_n$ and $\Psi(g)_M = 0 \ \forall M \neq N$.

$$\text{Then } P^{(+)} \Psi(g) = \frac{1}{\sqrt{N!}} \tilde{c}(g_1) \cdots \tilde{c}(g_N) \Omega .$$

c) Suppose $(e_\ell)_{\ell \in I}$ forms an ONB for h. Let I_0 collect all finite sequences $\ell \in I^N, N < \infty$, then "$\sim$" defined by

$$\ell \sim \ell' :\Leftrightarrow |\ell| = |\ell'| \text{ and } \exists \pi \in S_{|\ell|} \text{ s.t. } \ell_n = \ell'_{\pi(n)} \ \forall n$$

is an equivalence relation on I_0. Set $I^{(+)} := I_0/\sim$, and define $e(\ell) = \tilde{c}(e_{\ell_1}) \cdots \tilde{c}(e_{\ell_N}) \Omega \in \mathcal{F}^{(+)} \ \forall \ell \in I^{(+)}$.

Then $e(\ell)$ does not depend on the choice of representative in I_0, and $(e(\ell))_{\ell \in I^{(+)}}$ forms a complete orthogonal set in $\mathcal{F}^{(+)}$. (The set is not normalized.)

d) For every $\tilde{\Psi}, \Phi \in D_+^0$ and $g \in h$

$$(\Phi, \tilde{a}(g)\tilde{\Psi}) = (\tilde{c}(g)\Phi, \tilde{\Psi})$$
and $$(\Phi, \tilde{c}(g)\tilde{\Psi}) = (\tilde{a}(g)\Phi, \tilde{\Psi}).$$

e) If $f, g \in h$ then the following "canonical commutations relations" hold: $[A,B] := AB - BA$,

$$[\tilde{a}(f), \tilde{a}(g)] = 0|_{D_+^0} = [\tilde{c}(f), \tilde{c}(g)]$$

and $$[\tilde{a}(f), \tilde{c}(g)] = (f,g)_h \, \mathbb{1}|_{D_+^0}$$

Proof: If $\tilde{\Psi} \in \mathcal{F}^{(+)}$ is such that $\tilde{\Psi}_N = 0 \ \forall N \geq N_0$,
then $(a\tilde{\Psi})_N = 0 \ \forall N \geq N_0 - 1$ and $(c\tilde{\Psi})_N = 0 \ \forall N > N_0$.
$\Rightarrow \tilde{a}(g)\tilde{\Psi} \in D_+^0$ and $\tilde{c}(g)\tilde{\Psi} \in D_+^0$.
Hence, by an easy induction, "a)" holds. & "c)" follows from def.
"b)" is a consequence of Lemma 12.4.1.5. and "a)".
The proof of "d)" follows the outline of the proof of 12.4.1.2.
For $\ell \in I_0$, denote $\tilde{e}(\ell) := \tilde{\Psi}(e_{\ell_1}, \dots, e_{\ell_N})$, $N = |\ell|$.
These form an ONB for $\mathcal{F}_N^{(0)}$, and $e(\ell) = \sqrt{N!}\, P_N^{(+)} \tilde{e}(\ell) \ \forall \ell$.
Thus $(e(\ell), e(\ell')) = N! \left(\bigotimes_{n=1}^N e_{\ell_n}, P_N^{(+)} \left(\bigotimes_{n=1}^N e_{\ell'_n}\right)\right) \mathbb{1}(|\ell| = |\ell'|)$

$$= \mathbb{1}(|\ell| = |\ell'|) \sum_{\pi \in S_N} \prod_{n=1}^N (e_{\ell_n}, e_{\ell'_{\pi(n)}})$$

Thus if $\ell' \sim \ell'' \Rightarrow (e(\ell), e(\ell'')) = (e(\ell), e(\ell')) \ \forall \ell \in I_0$.
Also, if $\ell \not\sim \ell'$ then either $|\ell| \neq |\ell'|$ or $\forall \pi \in S_N \ \exists n$ s.t.
$\ell_n \neq \ell'_{\pi(n)} \Rightarrow (e_{\ell_n}, e_{\ell'_{\pi(n)}}) = 0 \Rightarrow (e(\ell), e(\ell')) = 0$.
If $\ell \sim \ell' \Rightarrow \exists \pi'$ s.t. $(e(\ell), e(\ell')) \stackrel{!}{=} \sum_{\pi \in S_N} \underbrace{\prod_{n=1}^N (e_{\ell'_{\pi'(n)}}, e_{\ell'_{\pi(n)}})}$

$\pi_0 = \pi \circ \pi'^{-1}$

$$\stackrel{!}{=} \sum_{\pi_0 \in S_N} \prod_{n=1}^N (e_{\ell'_n}, e_{\ell'_{\pi_0(n)}}) \qquad = \prod_{n=1}^N (e_{\ell'_n}, e_{\ell'_{\pi(\pi'^{-1}(n))}})$$

≥ 1 since each term is ≥ 0 and the term $\pi_0 = id$ yields 1. Hence, $(e(\ell), e(\ell')) = \mathbb{1}(\ell \sim \ell')\, \alpha_\ell$ where $\alpha_\ell \geq 1$ and $\alpha_\ell = \alpha_{\ell'}$ if $\ell \sim \ell'$. In fact, if $\ell \sim \ell'$, then

$$\sum_{\pi \in S_N} \bigotimes_{n=1}^N e_{\ell_{\pi(n)}} = \sum_{\pi \in S_N} \bigotimes_{n=1}^N e_{\ell'_{\pi'(\pi(n))}} = \sum_{\pi_0 \in S_N} \bigotimes_{n=1}^N e_{\ell'_{\pi_0(n)}}$$

and thus $e(\ell) = e(\ell')$ and it does not depend on the choice of representative. Thus $(e(\ell))_{\ell \in I^{(+)}}$ is an orthogonal set.

Now if $\Phi, \Psi \in \mathcal{F}^{(+)} \Rightarrow$

$$(\Phi, \Psi)_{\mathcal{F}^{(+)}} = (\Phi, \Psi)_{\mathcal{F}^{(0)}} = \sum_{\ell \in I_0} (\Phi, \tilde{e}(\ell))(\tilde{e}(\ell), \Psi)$$

and $(\tilde{e}(\ell), \Psi) = (\tilde{e}(\ell), P^{(+)}\Psi) = (P^{(+)}\tilde{e}(\ell), \Psi)$
$= \frac{1}{\sqrt{N!}} (e(\ell), \Psi) \Rightarrow$

$$(\Phi, \Psi) = \sum_{\ell \in I_0} \frac{1}{|\ell|!} (\Phi, e(\ell))(e(\ell), \Psi) = \sum_{\ell \in I^{(+)}} (\Phi, e(\ell))(e(\ell), \Psi)$$

$$* \sum_{\ell' \sim \ell} \frac{1}{|\ell|!} \le 1 \quad \forall \ell \in I_0 .$$

Thus $(e(\ell))_{\ell \in I^{(+)}}$ is also complete.

"d) & e)" Fix $f, g \in \mathfrak{h}$ and construct an ONB for $\mathfrak{h}$ as in the proof of 12.4.1.4. If $g = 0$, $a(g) = 0$ and $c(g) = 0$, and thus $\tilde{a}(g) = 0 = \tilde{c}(g)$ (on their domains). $\Rightarrow$ "d" holds. If $f = 0$ or $g = 0$, also $b(f) b'(g) = 0|_{D_+^0}$ for any choice of $b, b' = \tilde{a}, \tilde{c}$, and $(f, g)_{\mathfrak{h}} = 0$. Hence, "e" holds then.

Thus we can assume $f, g \neq 0$, and $e_0 = \frac{1}{\|f\|} f$.

Then, for any $\ell', \ell \in I^{(+)}$ with $N' := |\ell'| > 0$, $N := |\ell| \ge 0$

$$(e(\ell'), \tilde{c}(f) e(\ell)) = (P^{(+)} e(\ell'), c(f) e(\ell))$$
$$= \sqrt{N'! N!} \Big(P^{(+)}_{N'} \big(\bigotimes_{n=1}^{N'} e_{\ell'_n} \big), \frac{1}{N!} \sum_{\pi \in S_N} \sqrt{N+1} \, f \otimes \big(\bigotimes_{n=1}^{N} e_{\ell_{\pi(n)}} \big) \Big) \mathbb{1}(N' = N+1)$$
$$= \mathbb{1}(N' = N+1) \cancel{(N+1)!} \frac{1}{N!} \sum_{\pi \in S_N} \|f\| \frac{1}{\cancel{(N+1)!}} \sum_{\pi' \in S_{N+1}} \prod_{n=1}^{N} (e_{\ell'_{\pi'(n+1)}}, e_{\ell_{\pi(n)}}) \times (e_{\ell'_{\pi'(1)}}, e_0)$$

And $(e(\ell), \tilde{a}(f) e(\ell')) = (P^{(+)} e(\ell), a(f) e(\ell'))$

$$= \sqrt{N! N'!} \Big(P^{(+)}_N \big(\bigotimes_{n=1}^{N} e_{\ell_n} \big), \frac{1}{N'!} \sum_{\pi' \in S_{N'}} \sqrt{N'} (f, e_{\ell'_{\pi'(1)}}) \bigotimes_{n=2}^{N'} e_{\ell'_{\pi'(n)}} \Big) \mathbb{1}(N = N'-1)$$
$$= \mathbb{1}(N' = N+1) \frac{1}{N!} \sum_{\pi \in S_N} \sum_{\pi' \in S_{N+1}} \|f\| (e_0, e_{\ell'_{\pi'(1)}}) \prod_{n=1}^{N} (e_{\ell_{\pi(n)}}, e_{\ell'_{\pi'(n+1)}})$$
$$= (e(\ell'), \tilde{c}(f) e(\ell))^* = (\tilde{c}(f) e(\ell), e(\ell')). \; (*) \to p. 144$$

Thus for any $\Psi, \Phi \in D_+^0$, with $\gamma_\ell := \sum_{\ell' \sim \ell} \frac{1}{|\ell|!}$

$$(\Phi, \tilde{a}(f) \Psi) = \sum_{\ell' \in I^{(+)}} \gamma_{\ell'} (\Phi, e(\ell')) \Big[\sum_{\ell \in I^{(+)}} \gamma_\ell (e(\ell), \Psi) \times \underbrace{(e(\ell'), \tilde{a}(f) e(\ell))}_{= (\tilde{c}(f) e(\ell'), e(\ell))} \Big]$$
$$= \sum_{\ell' \in I^{(+)}} \gamma_{\ell'} (\Phi, e(\ell')) (\tilde{c}(f) e(\ell'), \Psi)$$

$= (\tilde{c}(f) \Phi, \Psi)$, where all sums have only finitely many non-zero terms. $\Rightarrow (\tilde{a}(f) \Psi, \Phi) = (\Psi, \tilde{c}(f) \Phi)$ (conjugation).

(*) Note that if $n' = 0$, then $\tilde{a}(f)e(\ell') = \tilde{a}(f)\Omega = 0$
and $(\tilde{c}(f)e(\ell), e(\ell')) = (\underbrace{(\tilde{c}(f)e(\ell))_0}_{=0}, \Omega) = 0$.
Thus also then

$$(e(\ell), \tilde{a}(f)e(\ell')) = (\tilde{c}(f)e(\ell), e(\ell')).$$

("e)") In the chosen basis, $\tilde{c}(g) = (e_0, g)\,\tilde{c}(e_0) + (e_1, g)\,\tilde{c}(e_1)$ and $\tilde{a}(g) = (g, e_0)\,\tilde{a}(e_0) + (g, e_1)\,\tilde{a}(e_1)$, $\tilde{c}(f) = \|f\|\,\tilde{c}(e_0)$ and $\tilde{a}(f) = \|f\|\,\tilde{a}(e_0)$. Thus for any $\ell \in I^{(+)}$, $\tilde{c}(g)e(\ell) = (e_0, g)\,e(\ell^{(0)}) + (e_1, g)\,e(\ell^{(1)})$ where $\ell^{(0)} := (0, \ell)$ and $\ell^{(1)} := (1, \ell)$, if $e_1 \neq 0$, and $\ell^{(1)} := \ell$ if $e_1 = 0$. (then the 2nd term is anyway zero.)

$\Rightarrow$ $\tilde{c}(f)\tilde{c}(g)e(\ell) = (e_0, g)\,e((0,0,\ell)) + (e_1, g)\,e((0, \ell^{(1)}))$
and $\tilde{c}(g)\tilde{c}(f)e(\ell) = \tilde{c}(g)\,e((0,\ell)) = (e_0, g)\,e((0,0,\ell)) + (e_1, g)\,e(\ell^{(2)})$
where $\ell^{(2)} = (0, \ell)$ if $e_1 = 0$ and $= (1, 0, \ell)$ if $e_1 \neq 0 \Rightarrow$
$e(\ell^{(2)}) = e((0,1,\ell)) = e((0, \ell^{(1)}))$. Thus

$$(\tilde{c}(f)\tilde{c}(g) - \tilde{c}(g)\tilde{c}(f))e(\ell) = 0 \quad \forall \ell \Rightarrow [\tilde{c}(f), \tilde{c}(g)] = 0.$$

Hence, if $\ell', \ell \in I^{(+)}$, we also have

$$(e(\ell'), \tilde{a}(f)\tilde{a}(g)e(\ell)) = (\tilde{c}(g)\tilde{c}(f)e(\ell'), e(\ell))$$
$$= (\tilde{c}(f)\tilde{c}(g)e(\ell'), e(\ell)) = (e(\ell'), \tilde{a}(g)\tilde{a}(f)e(\ell)).$$

$\Rightarrow$ $[\tilde{a}(f), \tilde{a}(g)] = 0$.

For $l \in I^{(+)}$, $N := |l| > 0$, we have $\tilde{a}(f)e(l)$
$= \underline{P}^{(+)}(0, \dots, \underbrace{\psi}_{N-1}, 0, \dots)$,

$$\psi = \frac{1}{\sqrt{N!}} \sum_{\pi \in S_N} a_N(f)\Big(\bigotimes_{n=1}^{N} e_{l_{\pi(n)}}\Big)$$

$$= \sqrt{N}\,\|f\| \frac{1}{\sqrt{N!}} \sum_{\pi \in S_N} (e_0, e_{l_{\pi(1)}}) \bigotimes_{n=2}^{N} e_{l_{\pi(n)}}$$

$$= \|f\| \frac{1}{\sqrt{(N-1)!}} \sum_{k=1}^{N} \mathbb{1}(l_k = 0) \sum_{\pi \in S_N} \mathbb{1}(\pi(1) = k) \bigotimes_{n=2}^{N} e_{l_{\pi(n)}}$$

$$\Rightarrow \underline{P}^{(+)}(0, \dots, \psi, 0, \dots)$$
$$= \frac{\|f\|}{\sqrt{(N-1)!}} \sum_{k=1}^{N} \mathbb{1}(l_k = 0) \underbrace{\sum_{\pi \in S_N} \mathbb{1}(\pi(1)=k)}_{=(N-1)!} \frac{e(\hat{l}^{(k)})}{\sqrt{(N-1)!}} \quad \Big| \; \hat{l}^{(k)} = \text{"} l \text{ omit } l_k \text{"}$$

$$\Rightarrow \tilde{a}(f)e(l) = \|f\| \sum_{k=1}^{N} \mathbb{1}(l_k = 0)\, e(\hat{l}^{(k)})$$

$$\Rightarrow \tilde{c}(g)\tilde{a}(f)e(l) = \sum_{k=1}^{N} \mathbb{1}(l_k = 0)\, \|f\| \Big[\overbrace{(e_0, g)\, e((0, \hat{l}^{(k)}))}^{= e(l), \text{ since } l_k = 0} + (e_1, g)\, e((\hat{l}^{(k)})^{(1)}) \Big]$$

$$= (f,g) \Big(\sum_{k=1}^{N} \mathbb{1}(l_k = 0)\Big) e(l) + \begin{cases} 0, & \text{if } e_1 = 0 \\ \|f\|\,(e_1, g) \sum_{k=1}^{N} \mathbb{1}(l_k = 0)\, e((1, \hat{l}^{(k)})), & \text{if } e_1 \neq 0 \end{cases}$$

Similarly, $\tilde{a}(f)\tilde{c}(g)e(l) = (e_0, g)\, \tilde{a}(f)\, e((0, l))$
$+ (e_1, g)\, \tilde{a}(f)\, e(\hat{e}^{(1)})$
where the first term $= (f, g) \frac{\|f\|}{\|f\|} \Big(e(l) + \sum_{k=1}^{N} \mathbb{1}(l_k = 0) e(0, \hat{l}^{(k)}) \Big)$

$$= (f,g) \Big(1 + \sum_{k=1}^{N} \mathbb{1}(l_k = 0)\Big) e(l)$$

and 2nd term, if $e_1 \neq 0$, $= (e_1, g)\, \|f\| \Big(0 + \sum_{k=1}^{N} \mathbb{1}(l_k = 0)\, e(((1,l)^{\wedge(k+1)}) \Big)$

$= (e_1, g)\, \|f\| \sum_{k=1}^{N} \mathbb{1}(l_k = 0)\, e((1, \hat{l}^{(k)}))$. If $e_1 = 0$, 2nd term $= 0$. Therefore,

$$(\tilde{a}(f)\tilde{c}(g) - \tilde{c}(g)\tilde{a}(f))e(l) = (f,g)\, e(l) \quad \forall l \in I^{(+)}.$$

$$\Rightarrow \tilde{a}(f)\tilde{c}(g) - \tilde{c}(g)\tilde{a}(f) = [\tilde{a}(f), \tilde{c}(g)] = (f,g)\, 1|_{D_+^0} \quad \square$$

By "d)" we have $\tilde{c}(g) \subset (\tilde{a}(g))^*$ and $\tilde{a}(g) \subset (\tilde{c}(g))^*$,
$\Rightarrow$ both operators are closable.$^{(*)}$ However, by "e)"
$[\tilde{a}(f), \tilde{a}(f)^*] \neq 0$ and $[\tilde{c}(g), \tilde{c}(g)^*] \neq 0$.
Hence, the closures are not normal operators.
For this reason, it is better to work with the family $\Phi_0(g)$, $g \in h$, whose domain is D_+^0 and there it is defined by

$$\Phi_0(g)\Psi = \frac{1}{\sqrt{2}}\left(\tilde{a}(g)\Psi + \tilde{c}(g)\Psi\right).$$

Then by "d)" $\forall\, \Phi, \Psi \in D_+^0$:

$$(\Phi, \Phi_0(g)\Psi) = \frac{1}{\sqrt{2}}\left[(\tilde{c}(g)\Phi, \Psi) + (\tilde{a}(g)\Phi, \Psi)\right]$$

$$= (\Phi_0(g)\Phi, \Psi) \Rightarrow \Phi_0(g) \text{ is densely def. and symmetric.}$$

In addition, defining $\Pi_0(g) := \Phi_0(ig)$, one finds

$$\frac{1}{\sqrt{2}}\left(\Phi_0(g) \pm i\Pi_0(g)\right) = \frac{1}{2}\left(\tilde{a}(g) + \tilde{c}(g) \pm i\left(-i\tilde{a}(g) + i\tilde{c}(g)\right)\right)$$

$$\Rightarrow \begin{cases} \tilde{a}(g) = \frac{1}{\sqrt{2}}\left(\Phi_0(g) + i\Pi_0(g)\right) \\ \tilde{c}(g) = \frac{1}{\sqrt{2}}\left(\Phi_0(g) - i\Pi_0(g)\right) \end{cases}$$

Thus it is possible to recover $\tilde{a}(g), \tilde{c}(g)$ from the family $\Phi_0(g)$, $g \in h$.

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(*) We set $c_+(g) := \overline{\tilde{c}(g)}$ and $a_+(g) := \overline{\tilde{a}(g)}$ and then
$c_+(g) = a_+(g)^* \Rightarrow c_+(g)^* = (a_+(g)^*)^* = \overline{a_+(g)} = a_+(g)$.
(Proof: We already have shown that $c_+(g) \subset \tilde{a}(g)^* \overset{5.7.}{=} a_+(g)^*$
If $\Phi \in D(\tilde{a}(g)^*)$ and $\Phi_0 := \tilde{a}(g)^*\Phi$, then $\forall\, \Psi \in D_+^0$:
$(\Phi_0, \Psi) = (\Phi, \tilde{a}(g)\Psi)$. Choose ONB s.t. $g = \|g\| e_0$,
$\Rightarrow e'(l) := \frac{1}{\|e(l)\|} e(l)$, $l \in I^{(+)}$, forms an ONB for $\mathcal{F}^{(+)}$.
Here $\nu_l^2 := \|e(l)\|^2 = \#\{\pi \in S_N \mid l_\pi = l\}$ (by p. 142) $\Rightarrow \nu_{(0,l)}^2 = \nu_l^2 (1 + \#\{n \mid l_n = 0\})$
Thus with $\beta_l := (e'(l), \Phi_0)$ we have $\Phi_0 = \sum_{l \in I^{(+)}} \beta_l e'(l)$.
If $0 \notin l$, $\beta_l = (\tilde{a}(g) e'(l), \Phi) = 0$, by the CCR. Thus, if we
def. $\Phi^{(N)} := \sum_{l \in I^{(+)}, |l| = N} \beta_{(0,l)} \frac{1}{\|g\|} \frac{\nu_l}{\nu_{(0,l)}} e'(l) \in D_+^0$, $\Phi^{(N)}$ is Cauchy, as $\frac{\nu_l}{\nu_{(0,l)}} \leq 1$.
Also $\tilde{c}(g)\Phi^{(N)} = \sum_{|l| \leq N+1} \beta_l e'(l) \to \Phi_0$. A combinatorial argument shows that $\Phi^{(N)} \to \Phi$.
$\Rightarrow \Phi \in D(c_+(g))$ and $\Phi_0 = c_+(g)\Phi$. $\square$)

Bratteli & Robinson: Operator algebras and quantum statistical mechanics, part 2, pp. 12 – 15. concern the basic properties of $\Phi(g) := \overline{\Phi_0(g)}$.

For instance, it is proven there that

2. Lemma: The symmetric operators $\Phi_0(g)$, $g \in h$, defined above satisfy:

(a) $\Phi_0(g)$ is essentially self-adjoint $\forall g \in h$
$\Rightarrow \Phi(g)$ is self-adjoint.

(b) If $g_n \in h$ are such that $g_n \to g$ in h, then
$\Phi_0(g_n)\Psi \to \Phi_0(g)\Psi$ in $\mathcal{F}^{(+)}$ $\forall \Psi \in D(\sqrt{\hat{N}}) \cap \mathcal{F}^{(+)}$.

(c)
$$\operatorname{span}\{\Phi_0(f_1)\Phi_0(f_2)\cdots\Phi_0(f_N)\Omega \mid N \in \mathbb{N}_0, f \in h^N\}$$
is dense in $\mathcal{F}^{(+)}$.

(d) $\forall f, g \in h$ and $\Psi \in D(\hat{N}) \cap \mathcal{F}^{(+)}$:

$$(\Phi_0(f)\Phi_0(g) - \Phi_0(g)\Phi_0(f))\Psi = i \operatorname{Im}(f,g)_h \Psi.$$

* Remarks: The proof of "(a)" relies on a theorem by Nelson: A closed symmetric operator A is self-adjoint iff $D(A)$ contains a dense set of 'analytic vectors' (see p. 51 for a definition). This is proven for instance in Theorem X.39 in Reed & Simon, part II.
* "c)" here is a corollary of 12.4.2.1.b proven above.

Hence, $\forall g \in h$ the operator $\Phi(g)$ is self-adjoint.
$\Rightarrow$ can define $W(g) := e^{i\Phi(g)}$ by spectral decomposition
$\Rightarrow$ $W(g)$ is unitary on $\mathcal{F}^{(+)}$. The operator $W(g)$ are called the Weyl operators and as shown in Bratteli & Robinson, the satisfy the Weyl form of the canonical commutation relations:

$$\forall f, g \in h: \; W(f)W(g) = e^{-i\frac{1}{2}\operatorname{Im}(f,g)} W(f+g)$$
$$= e^{-i\operatorname{Im}(f,g)} W(g)W(f)$$

This is a regular way to "extend" the CCR in 12.4.2.1.c) to closures.