GAUSSIAN FIELDS AND MULTIPLICATIVE CHAOS (spring 2017)

EXERCISE LIST

1. Recall the proof of Lemma 1.3. (use literature if needed): If probability measures μ_1, μ_2 on the probability space $(\Omega, \mathcal{F})$ agree on a π-system $\mathcal{T}$ that generates $\mathcal{F}$, then they are equal.

2. In order to show that Corollary 1.7 fails if one omits the assumption 'identically distributed', give an example of independent random variables $X_k \geq 0$ with $EX_k = 1$ for all $k \geq 1$ and such that $\sum_{k=1}^{\infty} X_k$ converges almost surely.

3. Prove in detail that the subset $\cap_{n=1}\infty \cup_{k=1}^{K(n)} \overline{B(x_k, 2^{-n})}$ of the Banach space E is compact (this set appears in the proof of Thm 2.1).

4. Let $X : (\Omega, \mathcal{F}) \to E$, where E is a separable Banach space. Show that X is measurable if it is weakly measurable, i.e. for any $\lambda \in E'$ the map $\omega \to \lambda(X(\omega))$ is measurable.

5. Let E be a separable Banach space and let X_1, X_2 be E-valued random variables defined on a common probability space. Show that the sum $X_1 + X_2$ is also a random variable (i.e. that it is measurable).

[Hint: Exercise 3.]

6. (i) Show in detail that under the conditions of Thm 3.4 one actually has exponential integrability $\mathbb{E}\, e^{\lambda \|Y\|} < \infty$ for *all* $\lambda > 0$.

(ii) In a similar way, in Kwapien's Thm 3.8 one has $\mathbb{E}\, e^{\lambda \|Y\|^2} < \infty$ for *all* $\lambda > 0$.

[Hint: prove this first for the tail sum $\sum_{k=m}^{\infty} \varepsilon_k u_k$ taking m large enough.]

7. Assume that $p : [0, \infty) \to [0, \infty)$ is strictly increasing, continuous, continuously differentiable on $(0, \infty)$ and $p(0) = 0$. Use Fubini to verify that for a random variable X we have

$$\mathbb{E}\, p(|X|) = \int_0^{\infty} p'(t) \mathbb{P}(|X| > \lambda).$$

8. Let Y, Y_k $(k \geq 1)$ be Gaussian real random variables such that $Y_k \xrightarrow{\mathbf{P}} Y$ in probability. Show that Y is also Gaussian.

9. Is the conclusion of the previous exercise true if Y and Y_k:s takes value in $\mathbf{R}^d$? Is it true if they take values in a separable Banach space?

10. Complete the proof of lemma 4.4 by verifying that $\mu_X = \mu$.

[Hint: ask the instructor for help if needed.]

11. Verify by using the form of the characteristic function that a Gaussian random variable on E (assumed to be separable) is symmetric if and only if $m_X := \mathbb{E}\, X = 0$.

12. Prove in detail by using just the definition of the Cameron-Martin norm that the Cameron-Martin space H_μ is a Banach space (only the completeness remains to be checked.)

13. Assume that E is a Banach space and $X : (\Omega, \mathcal{F}, d\mu) \to E$ is Bochner-integrable. Assume that the measurable subsets $A_j \in \mathcal{F}$ for $j = 1, 2, \ldots$ are disjoint. Prove that

$$\int_{\bigcup_{j=1}^\infty A_j} X d\mu = \sum_{j=1}^\infty \int_{A_j} X d\mu,$$

where the series on the right converges in E.

[Recall that one defines $\int_A X d\mu := \int_\Omega \chi_A(\omega) X(\omega) d\mu(\omega)$ for subsets $A \in \mathcal{F}$.]

14. Show that if E, F are Banach spaces, $X : (\Omega, \mathcal{F}, d\mu) \to E$ is Bochner-integrable, and $T : E \to F$ is linear and bounded operator, then $T(X)$ is also Bochner integrable and

$$T\left(\int_\Omega X d\mu\right) = \int_\omega T \circ F d\mu.$$

15. Equip $[0, 2\pi)$ with the Lebesgue σ-algebra and measure. Consider the ℓ^∞-valued map X, where

$$X(t) = (e^{it}, e^{2it}, e^{3it}, \ldots), \quad t \in [0, 2\pi).$$

(i) Is X weak*-measurable?

(ii) Is X strongly measurable?

(iii) Is X weakly measurable?

[Hint: for (iii)** ask the instructor for help if needed.]

16. Assume that E is a Banach space and $h_j \in E$ ($j \geq 1$). Let X_k:s be i.i.d. standard normals. Assume that the series

$$X = \sum_{k=1}^\infty X_k h_k$$

converges. Prove in detail that X is a centred Gaussian E-valued random variable.

17. Same assumptions than in Exercise 18. Denote by $\mu := \mu_X$ the distribution of X. Assume also that the linear span of vectors h_k is dense in E, and they are independent in the sense that for each j one has that $h_j \notin \overline{\text{span}}\{h_k : k \neq j\}$. Try to describe the Cameron-Maryin space H_μ in this situation!

18. Assume that H is a (separable) Hilbert space and μ is a Gaussian measure on H. Assume that $T : H \to H$ is an injective linear map such that the covariance operator of μ takes the form $C = TT^*$. Show that then $H_\mu = TH$ and $\|x\|_{H_\mu} = \|T^{-1}x\|_H$ for any $x \in H_\mu$.

19. Let $\mu = \mu_B$ be the distribution of the standard Brownian motion on $[0, 1]$ in the space $L^2(0, 1)$. Give a new simpler proof food the fact that $H_\mu = \{f \in W^{1,2}(0, 1) \ : \ f(0) = 1\}$ by using exercise 20 and the operator $T : L^2(0, 1) \to L^2(0, 1)$, where $Tf(x) = \int_0^x f(u)du$.

20. Assume that X is a Gaussian centred random variable on the Banach space E. Let E_1 be another Banach space, and let $A : E_1 \to E_2$ be a bounded linear operator.

(i) Check that AX is also Gaussian with values in E_2. How do you express the covariance operator of AX.

(ii) Assuming that A is injective on H_X prove that $H_{AX} = AH_X$ and $\|Au\|_{H_{AX}} = \|u\|_{H_X}$ for $u \in H_X$.

(ii) How do you formulate the result in (ii) if the injectivity assumption is dropped?

21. The Ornstein-Uhlenbeck process on $t \in [0, 1]$ may be defined as $X(t) = e^{-t/2}B(e^t)$, where B is the standard Brownian motion.

(i) Determine the covariance structure of the Ornstein-Uhlenback process on $t \in [0, 1]$.

(ii) Find the Cameron-Martin space of X.

[Hint: in (ii) you may apply the previous exercise.]

22. Try to use exercise 20 to compute the Cameron-Martin space of the Brownian bridge on $[0, 1]$ assuming that you know it for the standard Brownian motion on $[0, 1]$.

23. Use Theorem 4.22 of lectures to prove the Borell theorem:

Thm. (C. Borell) *Assume that μ is a centered Gaussian measure on the Banach space E. Let $A \subset E$ be a symmetric set ($A = -A$). Then for any $h \in H_\mu$ it holds that*

$$\mu(A + h) \geq \mu(A)e^{-\|h\|^2_{H_\mu}/2}.$$

[Suggestion: Note that $\mu(A + h) = \big(\mu(A + h) + \mu(A - h) = \mu_{-h}(A) + \mu_h(A)\big)$, and use Theorem 4.22 (ii) to estimate the last written expression.]

24. Use the previous exercise to give a lower bound (up to a constant) for the probability A_u as $u \to \infty$), where

$$A_u = \mathbb{P}\big(|B(t) - u\sin(\pi t)| \leq 1 \text{ for } 0 \leq t \leq 1\big).$$

25. Assume that $\rho : [0, \infty) \to [0, \infty)$ is increasing and continuous. If $\delta \in (0, 1)$, prove that

$$I := \int_0^\delta \sqrt{\log(1/u)}d\rho(u) < \infty$$

if and only if

$$\lim_{u\to 0^+} \sqrt{\log(1/u)}\rho(u) = 0 \qquad \text{and} \qquad \int_{\sqrt{\log(1/\delta)}}^\infty \rho(e^{-u^2}) < \infty.$$

Moreover, then it holds that

$$I = \rho(\delta)\sqrt{\log(1/\delta)} + \int_{\sqrt{\log(1/\delta)}}^\infty \rho(e^{-u^2}).$$

26. Let T be a compact metric space. Prove Dini's theorem: if the sequence of continuous functions $f_n : T \to \mathbf{R}$ converges pointwise monotonically on T to a continuous limit function, then the convergence is uniform.

27. Strengthen Theorem 3.3 of the lectures and prove that if E is a separable Banach space and X_n:s are E-valued, independent, and symmetric random variables such that the sum

$$\sum_{n=1}^{N} X_n$$

converges in distribution to an E-valued random variable X. Show that it then converges in probability, and hence by Theorem 3.3 also almost surely towards a limit random variable $\widetilde{X}$ with the same distribution as X.

[Suggestion: For $1 \leq n, m$ denote the distribution (measure) of the partial sum $\sum_{k=n+1}^{n+m} X_k$ by $\mu_{n,m}$, and denote by μ the distribution of X. Verify that the set $\{\mu_{n,m} : n, m \geq 1\}$ is tight. The convergence of the original series in probability is equivalent to $\mu_{n,m} \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$, or equivalently$\mu_{n,m} \xrightarrow{d} 0$. Assuming the contrary we may thus use tightness and pick subsequences n_ℓ, m_ℓ with $n_\ell \to \infty$, such that $n_\ell, n_\ell \xrightarrow{d} \nu \neq 0$ Deduce that $\mu * \nu = \mu$, or in other words, if Y is distributed as ν and independent of X we have $X \sim Y + X$. Show that this is impossible.]

28. Prove General Ito-Nishio's theorem (The original one dealt with uniform convergence of expansions to Brownian motion): Let E be a separable Banach space. Assume that X_n:s are E-valued, independent, and symmetric random variables such that for any $\lambda \in E'$ it holds that almost surely

$$\lambda\left(\sum_{n=1}^{N} X_n\right) \to \lambda(X),$$

where X is a E-valued random variable. Show that then the above series converges almost surely to X in the norm topology.

[Suggestion: Denote $S_N := \sum_{n=1}^{N} X_n$. Show first that $S_N \perp (X - S_N)$ for each N. Appy Lemma 4.16 of the lectures to deduce tightness of the partial sums $\{S_N\}$. Use Prohorov to prove that any subsequence of S_N has a further subsequence that converges in distribution, and then apply the condition of Ito-Nishio Theorem to verify that the limit has the same distribution as X. This yields convergence in distribution to X for the full series, and the rest follows from exercise 27.]

28. Show in detail how Theorem 6.10 (Slepian's inequality) follows in the general case as soon as it is knows in the case where T is finite.

29. Prove Corollary 6.11 of the lectures.

30. Show by a 2-dimensional counterexample that if one replaces $\sup X_t$ by $\sup |X_t|$ and $\sup Y_t$ by $\sup |Y_t|$ in Slepian's inequality (Thm. 6.10), then the obtained statement is false!

31. Try to find the Karhunen-Loewe expansion for the d-dimensional Brownian sheet X_s, which is the centered Gaussian field on $s \in [0,1]^d$ with the covariance structure

$$\mathbb{E}\, X_s X_u = \prod_{k=1}^{d} \min(s_k, u_k), \qquad s, u \in [0,1]^d.$$

32. Let μ be a centered Gaussian measure on a separable Banach space E. Prove that the linear operator $\widetilde{R}_\mu : L^2(d\mu) \to E$ is compact, where (as before)

$$\widetilde{R}_\mu \phi = \int_E x\phi(x)\mu(dx) \qquad \text{for} \quad \phi \in L^2(d\mu).$$

[Suggestion: Let $\varepsilon > 0$. Use Fernique's exponential tail for μ and Cauchy-Schwarz to verify that you get an $\varepsilon/2$-approximation in operator norm to $\widetilde{R}_\mu$ if the integration over E is replaced by just integration over the ball $B(0, r_0)$ and r_0 is taken large enough. Then, for any $\delta > 0$ use regularity of μ to pick a finite set $F \subset B(0, r_0)$ so that $\mu(B(0, r_0) \setminus (F + B(0, \delta))) < \delta$. For $x \in B(0, r_0)$ choose $g(x)$ be a measurable function such that $g(x) = 0$ if $x \notin F + B(0, \delta)$, and $|g(x) - x| \le \delta$ for $x \in F + B(0, \delta)$. Finally, check that the finite dimensional operator

$$\phi \mapsto \int_{B(0,r_0)} g(x)\phi(x)\mu(dx).$$

yields an ε-approximation in operator norm to $\widetilde{R}_\mu$ if δ is taken small enough.]

33. Let μ be a centered Gaussian measure on a separable Banach space E. Show that B_{H_μ} (the closed unit ball of the Cameron-Martin space) is a compact subset of E.

[Suggestion: After the previous exercise, it is enough (why) to verify the following functional analysis fact: Let $T : H \to E$ be a compact linear operator from a Hilbert space H to a Banach space E. Then the image of the unit ball, TB_H is a compact subset of E. Note, however that this is <u>not</u> true (can you find a counterexample?) even for one-dimensional operators if H is replaced by a general Banach space! If needed, ask for help in functional analysis from the instructor.]