GEOMETRIC MEASURE THEORY AND SINGULAR INTEGRALS: EXERCISE SET 3

Ask Emil Vuorinen (emil.vuorinen@helsinki.fi) for help! You can also ask after lectures. Exercise session is on Wednesday 1 March. **Hints are given on a separate file.**

In what follows we assume the following "baby $T1$" (i.e. a particularly simple $T1$) theorem to be known. Assuming only this, these exercises (together with the Set 2) prove a much more general $T1$ theorem (see Exercise 5).

0.1. **Theorem** (Baby $T1$)**.** *Let μ be a finite measure of order n, K be a standard n-dimensional kernel with $K(x,y) = -K(y,x)$, and $(T_\epsilon)_{\epsilon>0}$ be the corresponding SIO. Suppose that for some $\epsilon_0 > 0$ we have*

$$\|T_{\mu,\epsilon_0}1\|_{L^\infty(\mu)} \le C.$$

Then $T_{\mu,\epsilon_0}\colon L^2(\mu) \to L^2(\mu)$ boundedly.

(1) Let the setting be as in the Exercise 6 of the second set of exercises. Show that for all the large enough parameters $\lambda_0 > 0$ (depending only on the kernel estimates and such) we have

$$S \subset \{x \in \mathbb{R}^d \colon T_{\mu,*}1(x) > \lambda_0/2\}.$$

Next, suppose that

$$\int_{\mathbb{R}^d} |T_{\mu,*}1(x)|^{1/2}\, d\mu(x) \le C_0\mu(\mathbb{R}^d).$$

Conclude that now $\mu(\mathbb{R}^d \setminus S) \ge \mu(\mathbb{R}^d)/2$ if you fix λ_0 large enough depending on C_0. (Therefore, $\mathbb{R}^d \setminus S$ – the zero set of the 1-Lipschitz function $\Phi(x) = \operatorname{dist}(x, \mathbb{R}^d \setminus S)$ – is a big piece).

(2) Let μ be a finite measure of order n, K be a standard n-dimensional kernel with $K(x,y) = -K(y,x)$, and $(T_\epsilon)_{\epsilon>0}$ be the corresponding SIO. Suppose that

$$\int_{\mathbb{R}^d} |T_{\mu,*}1(x)|^{1/2}\, d\mu(x) \le C_0\mu(\mathbb{R}^d).$$

Show that there exists $G \subset \mathbb{R}^d$ such that $\mu(G) \ge \mu(\mathbb{R}^d)/2$ and $T_{\mu\lfloor G}\colon L^2(\mu\lfloor G) \to L^2(\mu\lfloor G)$ boundedly (i.e. this holds uniformly for $T_{\mu\lfloor G,\epsilon}$ with a bound depending on C_0, kernel estimates and such).

(3) Let μ be a measure of order n, K be a standard n-dimensional kernel with $K(x,y) = -K(y,x)$, and $(T_\epsilon)_{\epsilon>0}$ be the corresponding SIO. Suppose that

$$\|1_Q T_{\mu,*}1_Q\|_{L^{1/2}(\mu)}^{1/2} \le C_0\mu(2Q)$$

for all cubes $Q \subset \mathbb{R}^d$. Show that $T_\mu\colon L^2(\mu) \to L^2(\mu)$ boundedly (i.e. this holds uniformly for $T_{\mu,\epsilon}$).

(4) Let μ be a measure of order n, K be a standard n-dimensional kernel with $K(x,y) = -K(y,x)$, and $(T_\epsilon)_{\epsilon>0}$ be the corresponding SIO. Suppose that $\delta > 0$ and
$$\|1_{2Q} T_{\mu,\delta} 1_Q\|_{L^1(\mu)} \leq C_0 \mu(2Q)$$
for all cubes $Q \subset \mathbb{R}^d$. Show that
$$T_{\mu,*,\delta} 1_Q(x) \leq C + M_\mu^Q (1_{2Q} T_{\mu,\delta} 1_Q)(x)$$
for all cubes $Q \subset \mathbb{R}^d$ and $x \in Q$.

(5) Let μ be a measure of order n, K be a standard n-dimensional kernel with $K(x,y) = -K(y,x)$, and $(T_\epsilon)_{\epsilon>0}$ be the corresponding SIO. Suppose that
$$\sup_{\epsilon>0} \|1_{2Q} T_{\mu,\epsilon} 1_Q\|_{L^1(\mu)} \leq C_0 \mu(2Q)$$
for all cubes $Q \subset \mathbb{R}^d$. Show using Exercise 4 that
$$\|1_Q T_{\mu,*} 1_Q\|_{L^{1/2}(\mu)}^{1/2} \leq C \mu(2Q)$$
for all cubes $Q \subset \mathbb{R}^d$. In particular, $T_\mu \colon L^2(\mu) \to L^2(\mu)$ boundedly by Exercise 3. (Proving this with 1_{2Q} replaced with 1_Q in the assumptions is harder but can be done).