GEOMETRIC MEASURE THEORY AND SINGULAR INTEGRALS: EXERCISE SET 2

Ask Emil Vuorinen (emil.vuorinen@helsinki.fi) for help! You can also ask after lectures. They are not as hard or as long as they look!

The exercises are in a logical order so that the results of previous exercises are often needed. Exercise session is on Wednesday 15 February.

(1) Let $\Phi\colon \mathbb{R}^d \to [0,\infty)$ be 1-Lipschitz i.e. $|\Phi(x) - \Phi(y)| \le |x-y|$ for every $x, y \in \mathbb{R}^d$. Prove that given $\beta > 0$ and $x, y \in \mathbb{R}^d$ we have

$$|x-y|^{2\beta} + \Phi(x)^\beta \Phi(y)^\beta \ge c_\beta \Big[|x-y|^{2\beta} + \Phi(x)^{2\beta} + \Phi(y)^{2\beta} \Big]$$

for some $c_\beta > 0$ depending only on β.
Hint: consider the case $|x-y| > \frac{1}{2}\max(\Phi(x), \Phi(y))$ separately.

(2) Let K be a standard n-dimensional kernel and let $\Phi\colon \mathbb{R}^d \to [0,\infty)$ be 1-Lipschitz. For $\beta = \beta(n) = \max(1, n/2)$ define the suppression function

$$A_\Phi(x,y) = \frac{|x-y|^{2\beta}}{|x-y|^{2\beta} + \Phi(x)^\beta \Phi(y)^\beta} \in [0,1]$$

and the suppressed kernel

$$K_\Phi(x,y) = A_\Phi(x,y) K(x,y).$$

Show that in this case K_Φ is a standard n-dimensional kernel with bounds independent of Φ, and that in fact the size estimate holds in the stronger form

$$|K_\Phi(x,y)| \lesssim \frac{1}{(|x-y| + \Phi(x) + \Phi(y))^n}.$$

(So K_Φ behaves somewhat better than K (it suppresses K) but agrees with K when $\Phi(x) = 0$ or $\Phi(y) = 0$.)
Hint: When verifying the Hölder estimate you end up estimating $|A_\Phi(x,y) - A_\Phi(x',y)|$. This can be done by a direct calculation, or alternatively by writing

$$A_\Phi(x,y) - A_\Phi(x',y) = \int_0^1 \frac{d}{dt} A_\Phi(\gamma(t), y)\, dt,$$

where $\gamma(t) = x' + t(x - x')$ (think why this formula is valid), and estimating the derivative.

(3) Let μ be a measure of order n, K be a standard n-dimensional kernel and $\Phi\colon \mathbb{R}^d \to [0,\infty)$ be 1-Lipschitz. Define the kernel $K_\Phi(x,y)$ as in Exercise 2. Define the

natural notation (the Φ-suppressed singular integrals)

$$\begin{aligned} T_{\mu,\Phi,\epsilon}f(x) &= \int_{|x-y|>\epsilon} K_\Phi(x,y)f(y)\,d\mu(y); \\ T_{\mu,\Phi,*,\delta}f(x) &= \sup_{\epsilon>\delta} |T_{\mu,\Phi,\epsilon}f(x)|; \\ T_{\mu,\Phi,*}f(x) &= \sup_{\epsilon>0} |T_{\mu,\Phi,\epsilon}f(x)|. \end{aligned}$$

Show that

$$T_{\mu,\Phi,*}f(x) \le T_{\mu,\Phi,*,\Phi(x)}f(x) + CM_\mu f(x).$$

(4) Let the setting be exactly as in Exercise 3. Show that

$$T_{\mu,\Phi,*,\Phi(x)}f(x) \le T_{\mu,*,\Phi(x)}f(x) + CM_\mu f(x),$$

where you should recall that $T_{\mu,*,\Phi(x)}f(x) = \sup_{\epsilon>\Phi(x)} |T_{\mu,\epsilon}f(x)|$.

Conclude that the estimate from Exercise 3 can be improved to read

$$T_{\mu,\Phi,*}f(x) \le T_{\mu,*,\Phi(x)}f(x) + CM_\mu f(x).$$

You need this inequality in the exercises below.
Hint: Begin by writing $K_\Phi(x,y) = [K_\Phi(x,y) - K(x,y)] + K(x,y)$ so that it is enough to show that

$$\int_{|x-y|>\epsilon} |K_\Phi(x,y) - K(x,y)||f(y)|\,d\mu(y) \lesssim M_\mu f(x)$$

for every $\epsilon > \Phi(x)$.

(5) Suppose μ is a finite measure of order n and K is a standard n-dimensional kernel. Suppose $\lambda_0 > 0$ and define

$$S_0 = \{x\colon\ T_{\mu,*}1(x) > \lambda_0\}.$$

(Notice that we can hit the function 1 as the measure μ is finite). Show that with all 1-Lipschitz functions $\Phi\colon \mathbb{R}^d \to [0,\infty)$ we have

$$T_{\mu,\Phi,*}1(x) \le \lambda_0 + C, \qquad x \in \mathbb{R}^d \setminus S_0.$$

(6) Let the setting be exactly as in Exercise 5. Define

$$\epsilon(x) = \sup\{\epsilon > 0\colon\ |T_{\mu,\epsilon}1(x)| > \lambda_0\}.$$

Prove that $\epsilon(x) \in (0,\infty)$ for $x \in S_0$, and then define

$$S = \bigcup_{x\in S_0} B(x,\epsilon(x)).$$

Suppose Φ is any 1-Lipschitz function satisfying that $\Phi(x) \ge \operatorname{dist}(x, \mathbb{R}^d \setminus S)$ (e.g. $\Phi(x) = \operatorname{dist}(x,\mathbb{R}^d\setminus S)$). Show that

$$T_{\mu,\Phi,*}1(x) \le \lambda_0 + C, \qquad x \in \mathbb{R}^d.$$

(So $T_{\mu,\Phi,*}1 \in L^\infty(\mu)$ while we only had $1_{\mathbb{R}^d\setminus S_0}(x)T_{\mu,*}1(x) \le \lambda_0$! This is the point of suppression. Moreover, the suppressed and the original singular integrals agree in the set $\{\Phi = 0\}$, which one can in practical situations often arrange to satisfy $\mu(\{\Phi = 0\}) \sim \mu(\mathbb{R}^d)$ if λ_0 is large enough.)
Hint: recall that exercise 5 tells you that it is enough to show this for $x \in S_0$.