EXERCISES FOR APRIL 12

Exercises 1-3 are worth one point, and Exercise 4 is worth two points. For tips, ask Laura.

Recall that (on our course) a set $E \subset \mathbb{R}^2$ is called *uniformly* 1*-rectifiable*, if for every disc $B \subset \mathbb{R}^2$, the intersection $B \cap E$ can be covered by a continuum Γ_B with length $\mathcal{H}^1(\Gamma_B) \leq C \operatorname{diam}(B)$.

Exercise 1. Let $E \subset \mathbb{R}^2$ be a compact uniformly 1-rectifiable set. Prove that E can be covered by an AD regular continuum Γ with $\mathcal{H}^1(\Gamma) \lesssim \operatorname{diam}(E)$. The regularity constants will depend on the "C" in the definition of uniform 1-rectifiability.

Let μ, ν be Radon measures on $\mathbb{R}^n$, and let $x \in \operatorname{spt} \mu$. Recall ν is a *tangent measure* μ *at* x, if $\nu \neq 0$, and there exist sequences $(c_i)_{i \in \mathbb{N}}, (r_i)_{i \in \mathbb{N}}$ of positive reals, with $r_i \searrow 0$, such that

$$c_i T_{x,r_i \sharp} \mu \to \nu$$

weakly. Here $T_{x,r}$ is the affine mapping $T_{x,r}(y) = (y - x)/r$, which takes the ball $B(x,r)$ to the ball $B(0,1)$. Weak convergence, say $\mu_i \to \mu$, just means that

$$\int \psi \, d\mu_i \to \int \psi \, d\mu$$

for all compactly supported continuous functions $\psi \colon \mathbb{R}^n \to \mathbb{R}$. The family of all tangent measures of μ at x is denoted by $\operatorname{Tan}(\mu, x)$. For more information about tangent measures, see Pertti's book, Chapter 14.

Exercise 2. Check rigorously the following (variant of a) lemma, which Hans used in his presentation. Let μ be a Radon measure on $\mathbb{R}^n$, and let $\varphi \colon \operatorname{spt} \mu \to (0, \infty)$ be a continuous function (note that φ is required to be strictly positive). Consider the measure $\lambda = \varphi \, d\mu$, defined by

$$\lambda(B) = \int_B \varphi \, d\mu$$

for Borel sets B. Prove that $\operatorname{Tan}(\mu, x) = \operatorname{Tan}(\lambda, x)$ for all $x \in \operatorname{spt} \mu$.

The next exercises, borrowed from Pertti's book, are an introduction to the concept of *removability*, which we will discuss more (and Janne will talk about) after the Easter break. A compact set $E \subset \mathbb{C} \cong \mathbb{R}^2$ is called *removable*, if every bounded analytic function $f \colon \mathbb{C} \setminus E \to \mathbb{C}$ can be extended to an analytic function $\tilde{f} \colon \mathbb{C} \to \mathbb{C}$ (such functions are called *entire*). As a basic example, all singletons are removable. The next exercise demonstrates that compact sets with dimension > 1 are **not** removable.

Exercise 3. Let μ be a Radon measure on $\mathbb{C}$ with $E = \operatorname{spt} \mu$ compact. Assume that, for some $1 < s < 2$ and $C \geq 1$, the measure μ satisfies the uniform growth bound $\mu(B(x,r)) \leq Cr^s$. Prove that the Cauchy transform

$$\mathcal{C}_\mu 1(z) = \int \frac{d\mu w}{z - w}, \qquad z \in \mathbb{C} \setminus E,$$

defines a **non-constant, bounded and** $(s-1)$**-Hölder continuous** analytic function on $\mathbb{C}\setminus E$. Deduce that E is not removable. *Hint*: To prove non-constancy, find $\lim_{z\to\infty} C_\mu 1(z)$ and $\lim_{z\to\infty} z \cdot C_\mu 1(z)$. For the last statement about non-removability, you may use Liouville's theorem, which states that a non-constant entire function cannot be bounded.

The case of 1-dimensional sets is delicate, and much of the theory presented in the course was motivated by the question: which 1-dimensional sets are removable?

Exercise 4 (Worth two points)**.** Let $E \subset [0,1]^2$ be the *four corners Cantor set*, depicted below, and let $\mu = \mathcal{H}^1|_E$. Prove that the Cauchy transform

$$\mathcal{C}_\mu 1(z) = \int \frac{d\mu w}{z-w}, \qquad z \in \mathbb{C} \setminus E$$

is **unbounded** on $\mathbb{C} \setminus E$. *Hint*: Consider points of the form $(-\epsilon, 0)$. You may use the fact that μ is 1-AD regular.

The point of the previous exercise is simply to demonstrate that the removability of the four corners Cantor set E is a non-trivial problem: the technique of the third exercise does not show that E would be non-removable. In fact, it turns out that E **is** removable. If time permits, we will prove this later on the course.

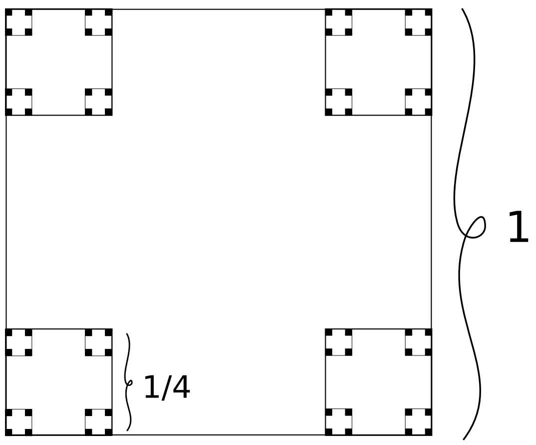

FIGURE 1. The four corners Cantor set (at least the first couple of iterations). The set E is obtained by iterating the scheme infinitely many times, and you may take for granted that $0 < \mathcal{H}^1(E) < \infty$.