On Well-Ordered Classes

We show every non-empty well-ordered class has a minimal element.

1. Suppose $\leq_A$ well-orders a class A, such that every subset $X \neq \emptyset$ such that $X \subseteq A$, has a $\leq_A$-smallest element in X.

2. Suppose $\phi(x, y)$ defines $\leq_A$ and $\psi(x)$ defines A.

3. Suppose ϕ and ψ have quantifier-rank at most n. (The quantifier rank of a formula is the number of alternations of blocks of existential and universal quantifiers.)

4. Fix $c \in A$.

5. Suppose κ is a cardinal such that $c \in H(\kappa) \prec_{n+1} V$. (We apply here the *Levy Reflection Principle*. $H(\kappa)$ is the collection of sets of hereditary cardinality κ.)

6. Let $B = \{b \in H(\kappa) : H(\kappa) \models \psi(b)\}$. Thus $B \subseteq A$.

7. Since B is a set, $B \neq \emptyset$ and $B \subseteq A$, there is $a \in B$ such that a is the $\leq_A$-smallest element of B.

8. CLAIM: a is the smallest element of A. Suppose there is $b \in A$ such that $b <_A a$. Thus $V \models \exists x(\psi(x, a) \wedge x \neq a)$.

9. By the choice of κ, $H(\kappa) \models \exists x(\psi(x) \wedge \phi(x, a) \wedge x \neq a)$. Let $b \in H(\kappa)$ such that $H(\kappa) \models \psi(b) \wedge \phi(b, a) \wedge b \neq a$. Note that $b \in B$.

10. By the choice of κ again, $V \models \psi(b) \wedge \phi(b, a) \wedge b \neq a$. This contradicts the choice of a as the $\leq_A$-smallest element of B, proving the CLAIM.

11. Question: Can one prove for any given subclass $C \neq \emptyset$ of A, that C has a smallest element?