HOMEWORK 4

(1) (10pts) Prove that $x^5 + 48x + 24$ is irreducible in $\mathbb{Q}[x]$.

(2) (20pts) Let $A = \mathbb{Z}[i]$.
  (a) The principle ideal (2) of A generated by 2 is not a prime ideal. Hint: Find $x, y \notin (2)$, but $xy \in (2)$.
  (b) The principle ideal (3) of A is a prime ideal. Hint: If $xy \in (3)$, consider the norm of xy, use divisibility of 3 to conclude that one of x, y has to be in (3). You need to consider what happens when you have 3 divides $a^2 + b^2$ with $a, b \in \mathbb{Z}$.
  (c) The principle ideal (5) of A is not a prime ideal.
  (d) The principle ideal (7) of A is a prime ideal.

(3) (10pts) Let α and β be distinct prime ideals of a Dedekind domain A. Show that $\alpha\beta = \alpha \cap \beta$. Hint: use Lemma 1 on Page 18 of the Book.

(4) (10pts) A: a ring. α: an ideal.
  (a) If β_1, β_2 are two ideals such that $\alpha \subset \beta_1 \cup \beta_2$, then $\alpha \subset \beta_1$ or $\alpha \subset \beta_2$.
  (b) Suppose $\gamma_1, \ldots, \gamma_n$ $(n \geq 2)$ are *prime* ideals, such that $\alpha \subset \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n$, then $\alpha \subset \gamma_i$ for one of the i. HINT: use induction on n. Think carefully about the case $n = 3$.

Addendum (NOT HW).

- You may wonder when is (p) a prime ideal in $\mathbb{Z}[i]$, and when it is not. If you try some more computations with $p = 11, 13, 17$ etc, you will quickly find some pattern. There is indeed a general criterion, using something called Legendre symbol.