HOMEWORK 5

(1) (10pts) A: Dedekind domain. α, β : ideals of A. Show that the following are equivalent:
   (a) $\alpha + \beta = A$
   (b) Write $\alpha = \prod p_i^{\alpha_i}$ and $\beta = \prod q_j^{\beta_j}$ with $\alpha_i > 0, \beta_j > 0$ the decomposition into product of prime ideals, then $p_i \neq q_j, \forall i, j$.

(2) (10pts) A: Dedekind domain. $\alpha \subset A$ an ideal. Suppose $\alpha = \prod_{i=1}^{n} p_i^{\alpha_i}$ where $\alpha_i > 0$ and $p_i \neq p_j, \forall i \neq j$. Show that there exists an isomorphism

$$A/\alpha \simeq \prod_{i=1}^{n} A/p_i^{\alpha_i}.$$

(3) (10pts) A: Dedekind domain. $\alpha, \beta, \gamma \subset A$ ideals. Suppose $p_1, p_2, p_3 \subset A$ are prime ideals. Suppose

$$\alpha = p_1 p_3^2, \beta = p_1^2 p_2^2 p_3^2, \gamma = p_1^5 p_3$$

the decomposition of ideals. What is the decomposition for $\alpha + \beta + \gamma$?

(4) (10pts) Let $A = \mathbb{Z}[\sqrt{-5}]$. Show that:
   (a) The ideal $(3, 1 + \sqrt{-5})$ is a prime ideal.
   (b) The ideal $(3, 1 - \sqrt{-5})$ is a prime ideal.
   (c) $(3, 1 + \sqrt{-5}) \cdot (3, 1 - \sqrt{-5}) = (3)$.