HOMEWORK 2

(1) (10pts). Is $\sqrt{3} + \sqrt[3]{7} + 1$ integral over $\mathbb{Z}$? Give reasons.

(2) (20pts). Let $f : A \hookrightarrow B$ be an injective morphism of rings, and regard A as a subring of B. Suppose B is integral over A. Let $\beta \subseteq B$ be an ideal of B. Define $\alpha := f^{-1}(\beta)$. Show that
   (a) α is an ideal of A.
   (b) If β is a prime ideal, then α is also a prime ideal.
   (c) The quotient ring B/β is integral over A/α.
   (d) Suppose β is a prime ideal. Then β is a maximal ideal if and only if α is a maximal ideal.

Definition: Let B be an integral domain, K its field of fractions. If for any nonzero $x \in K$, we have $x \in B$ or $x^{-1} \in B$, then we call B a *valuation ring* of K.

(3) (20pts) Let k be a field, $B = k[[x]]$ the power series ring. Let K be the fraction field of B.
   (a) If $f = a_0 + a_1 x + \ldots + a_n x^n + \ldots$ is an element of B. Show that f is a unit in B if and only if $a_0 \neq 0$.
   (b) Define $\mathfrak{m} = \{x \in B, x \text{ is not a unit of } B\}$. Show that $\mathfrak{m}$ is a maximal ideal.
   (c) What is K like?
   (d) Show that B is a valuation ring of K.

(4) (20pts). This exercise generalizes many things in the above exercise, to the general setting.
Let B be an integral domain, K its field of fractions. Suppose B is a valuation ring of K. Define $\mathfrak{m} = \{x \in B, x \text{ is not a unit of } B\}$.
   (a) Show that $\mathfrak{m}$ is an ideal.
   (b) Show that $\mathfrak{m}$ is in fact a maximal ideal.
   (c) Show that $\mathfrak{m}$ is the unique maximal ideal of B.
   (d) Show that B is integrally closed in K.